GRB

Problems in CALCULUS

For JEE (Main & Advanced)

& All Other Engineering Entrance Examinations

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Hints & Solutions of Problems in Calculus

By: Sameer Bansal

Sameer Bansal

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He has thirteen years teaching experience in JEE coaching and has taught various JEE toppers. He is known for his unique methods of solving problems. His teaching methodology is so simple that students feel mathematics very comfortable and interesting.

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Under the Guidance of

Mr. V.K. Bansal

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PREFACE

I am pleased to present this book before the readers. It is specially designed to help the students preparing for **JEE (Main & Advanced)** & all other engineering entrance examinations and strictly based upon the current pattern being followed in JEE. Each chapter includes different types of problems, which are :

(1) Key Concepts,

(2) Only One Correct Answer,

(3) Linked Comprehension Type,

(4) More Than One Correct Answers,

(5) Match the Columns Type,

(6) Integer Answer Type.

This book is complete solution for the students facing the problems in calculus. Here I have made an effort to provide the unique and latest pattern problems from my experienced teaching career.

I hope that this book will be more useful to the students and learned teachers. Suggestions for further improvement of the book will be gratefully acknowledged.

I would like to thank Shri Manoj Kumar Bathla, Proprietor of G. R. Bathla & Sons and Mr. Sugam Bathla for their sincere efforts in bringing the present edition in such a nice form.

Finally, I convey my best wishes to all aspirants for their studies and success in their career.

May, 2018 **Sameer Bansal**

Note : Students and honourable teachers may feel free to give valuable suggestions on the mail suggestionsgrb@gmail.com to improve the quality of book.

Acknowledgement

This book is dedicated to my dear father Mr. V.K. Bansal because he is the only person who inspired me to write this book and without the blessings and support of my father, it was impossible to complete this book.

My sincere thanks to all my colleague's students.

Here I would like to convey my affectionate thanks to my family especially my wife Mrs. Mahima Bansal, son Avighn and mother Mrs. Neelam Bansal. My wife motivated and supported me in every phase of this book writing and she has done the complete proof reading of this book.

I would also like to extend my special thanks to Mr. P.K. Bansal, for his support and guidance.

Compliments for the EDP editors of this book Mr. Rakesh Panchal and Mr. Arun Prasannan for completing the type setting of this book in a new style and format.

Sameer Bansal

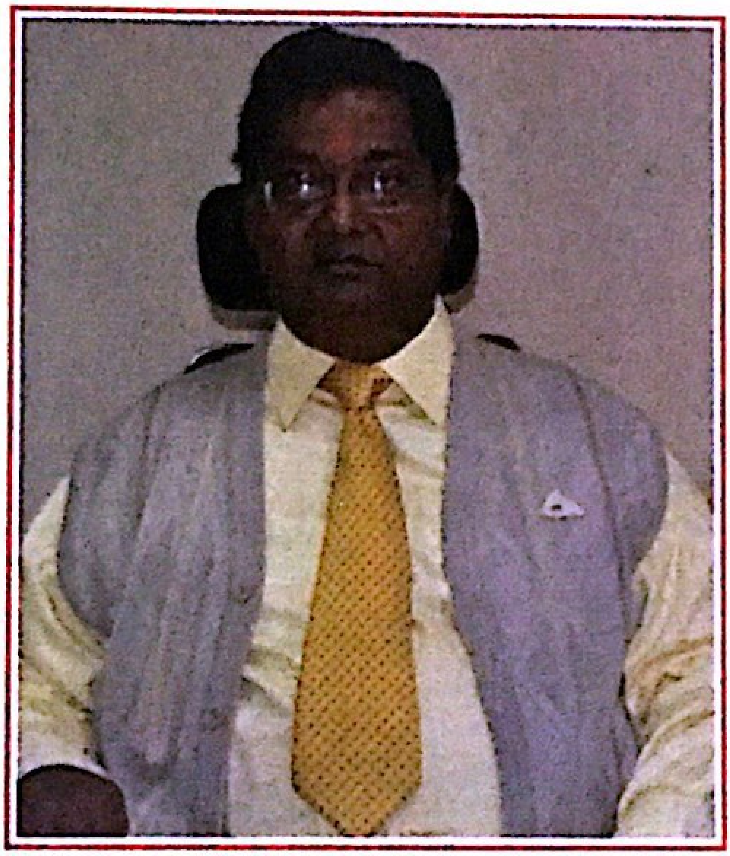
Dedication

Every word of this book is dedicated to all those students who are willing to achieve their goal of life.

I am deeply indebted to my father & esteemed teacher

Mr. V.K. Bansal

for his support and guidance.



Mr. V.K. Bansal

CONTENTS

Chapter 1

Functions

KEY CONCEPTS

1. GENERAL DEFINITION :

If to every value (considered as real unless otherwise stated) of a variable x, which belongs to some collection (Set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a function (Single valued) of x or a dependent variable defined on the set E ; x is the argument or independent variable .

If to every value of x belonging to some set E there corresponds one or several values of the variable y, then y is called a multiple valued function of x defined on E. Conventionally the word **"FUNCTION"** is used only as the meaning of a single valued function, if not otherwise stated.

Pictorially : $\xrightarrow[\text{input}]{x} \boxed{f} \xrightarrow[\text{output}]{f(x)=y}$, y is called the image of x and x is the pre-image of y under f.

Every function from $A \to B$ satisfies the following conditions .

(a) $f \subset A \times B$

(b) $\forall\, a \in A \Rightarrow (a, f(a)) \in f$ and

(c) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

2. DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION :

Let $f: A \to B$, then the set A is known as the domain of f and the set B is known as co-domain of f. The set of all f images of elements of A is known as the range of f. Thus :

Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

It should be noted that range is a subset of co-domain . If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

3. IMPORTANT TYPES OF FUNCTIONS :

1. Polynomial Function :

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note : (a) A polynomial of degree one with no constant term is called an odd linear function. *i.e.* $f(x) = ax$, $a \neq 0$

(b) There are two polynomial functions , satisfying the relation;
$f(x) . f(1/x) = f(x) + f(1/x)$. They are :
(i) $f(x) = x^n + 1$ and
(ii) $f(x) = 1 - x^n$, where n is a positive integer.

2. Algebraic Function :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form

$P_0(x)y^n + P_1(x)y^{n-1} + ... + P_{n-1}(x)y + P_n(x) = 0$ where n is a positive integer and $P_0(x), P_1(x)$ .... are polynomials in x.

e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called **Transcendental Function.**

3. Fractional Rational Function :

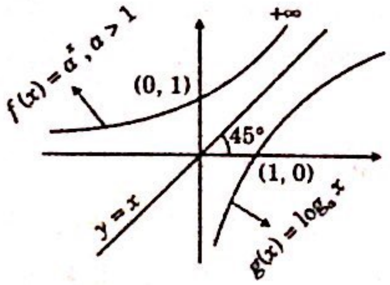
A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$,

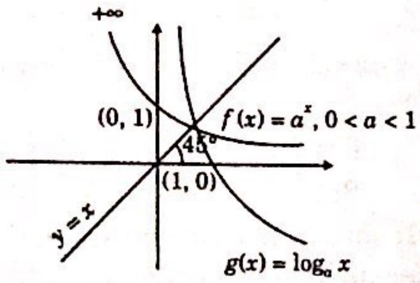
where $g(x)$ and $h(x)$ are polynomials and $h(x) \neq 0$.

4. Exponential Function :

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, *i.e.*, $g(x) = \log_a x$.

Note that $f(x)$ and $g(x)$ are inverse of each other and their graphs are as shown.





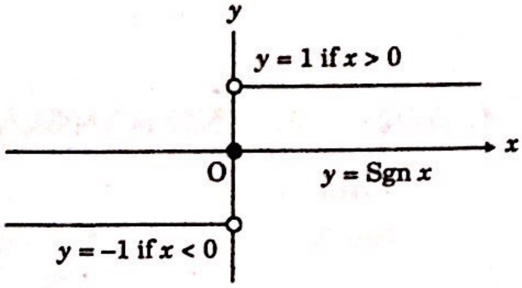
5. Absolute Value Function :

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

6. Signum Function :

A function $y = f(x) = \text{sgn}(x)$ is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



It is also written as $\text{sgn}\, x = |x|/x$; $x \neq 0$; $f(0) = 0$

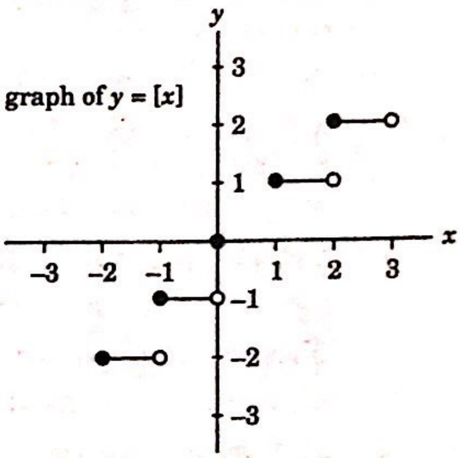
7. Greatest Integer or Step Up Function :

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x. Note that for :

$-1 \leq x < 0$; $[x] = -1$ $0 \leq x < 1$; $[x] = 0$

$1 \leq x < 2$; $[x] = 1$ $2 \leq x < 3$; $[x] = 2$

and so on .



Properties of greatest integer function :

(a) $[x] \leq x < [x] + 1$ and

$x - 1 < [x] \leq x$, $0 \leq x - [x] < 1$

(b) $[x + m] = [x] + m$ if m is an integer.

(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

(d) $[x] + [-x] = 0$ if x is an integer

$= -1$ otherwise.

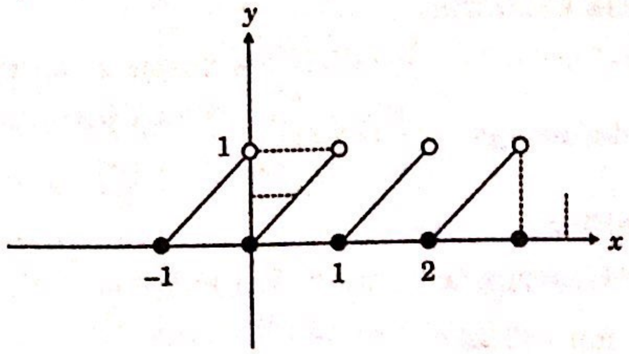
8. Fractional Part Function :

It is defined as :

$g(x) = \{x\} = x - [x]$.

e.g. the fractional part of the no. 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3

The period of this function is 1 and graph of this function is as shown.



4. DOMAINS AND RANGES OF COMMON FUNCTION :

Function ($y = f(x)$)	Domain (*i.e.* values taken by x)		Range (*i.e.* values taken by $f(x)$)	
A. Algebraic Functions				
(a) x^n, $(n \in N)$	R = (set of real numbers)		R,	if n is odd
			$R^+ \cup \{0\}$,	if n is even
(b) $\frac{1}{x^n}$, $(n \in N)$	$R - \{0\}$		$R - \{0\}$,	if n is odd
			R^+,	if n is even
(c) $x^{1/n}$, $(n \in N)$	R,	if n is odd	R ,	if n is odd
	$R^+ \cup \{0\}$,	if n is even	$R^+ \cup \{0\}$,	if n is even
(d) $\frac{1}{x^{1/n}}$, $(n \in N)$	$R - \{0\}$,	if n is odd	$R - \{0\}$,	if n is odd
	R^+ ,	if n is even	R^+ ,	if n is even
B. Trigonometric Functions				
(a) $\sin x$	R		$[-1, +1]$	
(b) $\cos x$	R		$[-1, +1]$	
(c) $\tan x$	$R - (2k+1)\frac{\pi}{2}, k \in I$		R	
(d) $\sec x$	$R - (2k+1)\frac{\pi}{2}, k \in I$		$(-\infty, -1] \cup [1, \infty)$	
(e) $\text{cosec}\, x$	$R - k\pi, k \in I$		$(-\infty, -1] \cup [1, \infty)$	
(f) $\cot x$	$R - k\pi, k \in I$		R	
C. Inverse Circular Functions (Refer after Inverse is taught)				
(a) $\sin^{-1} x$	$[-1, +1]$		$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	
(b) $\cos^{-1} x$	$[-1, +1]$		$[0, \pi]$	

(c) $\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(d) $\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(e) $\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
(f) $\cot^{-1} x$	R	$(0, \pi)$
D. Exponential Functions		
(a) e^x	R	R^+
(b) $e^{1/x}$	$R - \{0\}$	$R^+ - \{1\}$
(c) $a^x, a > 0$	R	R^+
(d) $a^{1/x}, a > 0$	$R - \{0\}$	$R^+ - \{1\}$
E. Logarithmic Functions		
(a) $\log_a x$, $(a > 0)$ $(a \neq 1)$	R^+	R
(b) $\log_x a = \frac{1}{\log_a x}$ $(a > 0)$ $(a \neq 1)$	$R^+ - \{1\}$	$R - \{0\}$
F. Integral Part Functions		
(a) $[x]$	R	I
(b) $\frac{1}{[x]}$	$R - [0, 1)$	$\left\{\frac{1}{n}, n \in I - \{0\}\right\}$
G. Fractional Part Functions		
(a) $\{x\}$	R	$[0, 1)$
(b) $\frac{1}{\{x\}}$	$R - I$	$(1, \infty)$
H. Modulus Functions		
(a) $\lvert x \rvert$	R	$R^+ \cup \{0\}$
(b) $\frac{1}{\lvert x \rvert}$	$R - \{0\}$	R^+
I. Signum Function		
$\text{sgn}(x) = \frac{\lvert x \rvert}{x}, x \neq 0$; $= 0, x = 0$	R	$\{-1, 0, 1\}$

J. Constant Function

say $f(x) = c$ R $\{c\}$

5. EQUAL OR IDENTICAL FUNCTION :

Two functions f and g are said to be equal if :

(a) The domain of f = the domain of g.

(b) The range of f = the range of g and

(c) $f(x) = g(x)$, for every x belonging to their common domain. *e.g.*,

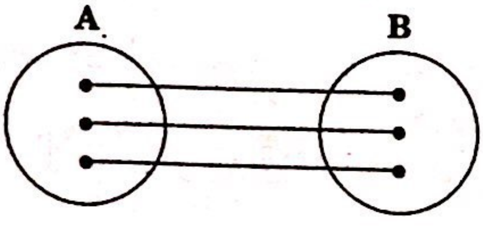
$f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ are identical functions.

6. CLASSIFICATION OF FUNCTIONS :

One-One Function (Injective Mapping) :

A function $f : A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ and $f(x_1)$, $f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an Injective Mapping can be shown as



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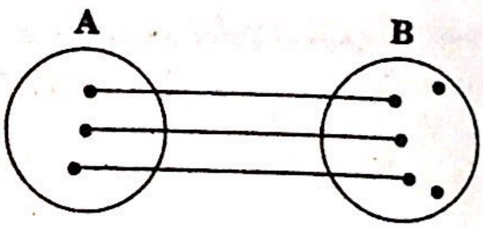


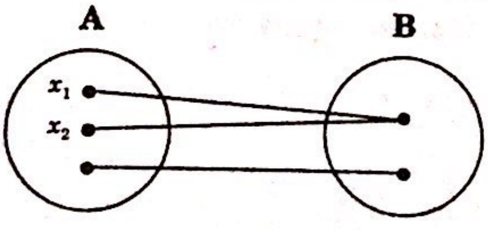
Note : (a) Any function which is entirely increasing or decreasing in whole domain, then $f(x)$ is one-one.

(b) If any line parallel to x-axis cuts the graph of the function atmost at one point, then the function is one-one.

Many-One Function :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B. Thus $f : A \to B$ is many one if for; $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a Many-One Mapping can be shown as



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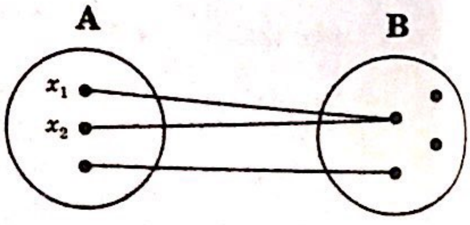


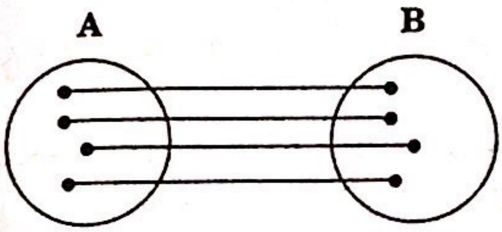
Note : (a) Any continuous function which has atleast one local maximum or local minimum, then $f(x)$ is many-one. In other words, if a line parallel to x-axis cuts the graph of the function atleast at two points, then f is many-one.

(b) If a function is one-one, it cannot be many-one and vice versa.

Onto Function (Surjective Mapping) :

If the function $f : A \to B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B . Thus $f : A \to B$ is surjective if $\forall\ b \in B, \exists$ some $a \in A$ such that $f(a) = b$.

Diagrammatically Surjective Mapping can be shown as



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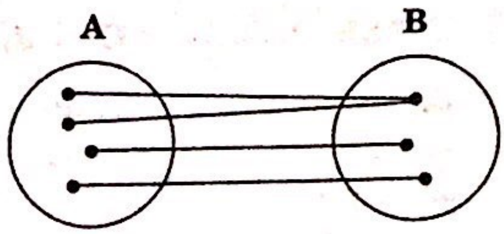
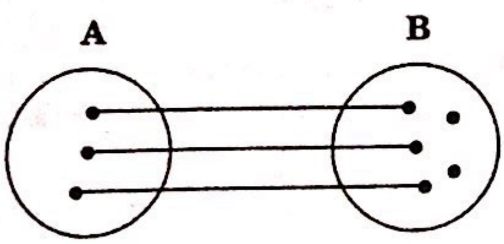


Note : If range = co-domain, then $f(x)$ is onto.

Into Function :

If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then $f(x)$ is into .

Diagrammatically Into Function can be shown as



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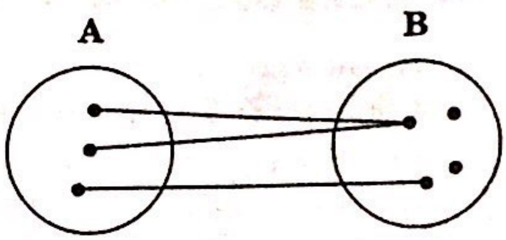


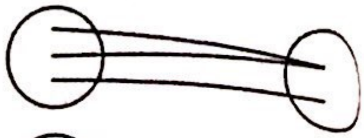
Note : If a function is onto, it cannot be into and vice versa. A polynomial of degree even will always be into.

Thus a function can be one of these four types :

(a) one-one onto (injective and surjective)

(b) one-one into (injective but not surjective)

(c) many-one onto (surjective but not injective)



(d) many-one into (neither surjective nor injective)

Note : (a) If f is both injective and surjective, then it is called a **Bijective** mapping.
The bijective functions are also named as invertible, non singular or biuniform functions.
(b) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n and out of it n ! are one-one.

Identity Function :

The function $f: A \to A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function is a bijection .

Constant Function :

A function $f: A \to B$ is said to be a constant function if every element of A has the same f image in B . Thus $f: A \to B$; $f(x) = c$, $\forall \ x \in A$, $c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into .

7. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f and g are real valued functions of x with domain set A, B respectively, then both f and g are defined in $A \cap B$. Now we define $f+g$, $f-g$, $(f.g)$ and (f/g) as follows :

(a) $(f \pm g)(x) = f(x) \pm g(x)$
(b) $(f.g)(x) = f(x).g(x)$
] domain in each case is $A \cap B$

(c) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B \ s.t \ g(x) \neq 0\}$.

8. COMPOSITE OF UNIFORMLY AND NON-UNIFORMLY DEFINED FUNCTIONS :

Let $f: A \to B$ and $g: B \to C$ be two functions. Then the function $gof: A \to C$ defined by $(gof)(x) = g(f(x)) \ \forall \ x \in A$ is called the composite of the two functions f and g.

Diagrammatically $\xrightarrow{x}$ $\boxed{f}$ $\xrightarrow{f(x)}$ $\boxed{g}$ $\longrightarrow$ $g(f(x))$.

Thus the image of every $x \in A$ under the function gof is the g-image of the f-image of x. Note that gof is defined only if $\forall \ x \in A, f(x)$ is an element of the domain of g so that we can take its g-image. Hence, for the product gof of two functions f and g, the range of f must be a subset of the domain of g.

Properties of Composite Functions :

(a) The composite of functions is not commutative, *i.e.*, $gof \neq fog$.

(b) The composite of functions is associative *i.e.*, if f, g, h are three functions such that $fo\,(goh)$ and $(fog)\,oh$ are defined, then $fo\,(goh) = (fog)\,oh$.

(c) The composite of two bijections is a bijection *i.e.*, if f and g are two bijections such that gof is defined, then gof is also a bijection.

9. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x and y. Symbolically if, $f(tx, ty) = t^n \cdot f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

10. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

11. IMPLICIT AND EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **Implicit Function**. For *e.g.* the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit Function.**

12. INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one-one and onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A$ and $y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties of Inverse Function :

(a) The inverse of a bijection is unique.

(b) If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B respectively. Note that the graphs of f and g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x', y') corresponding to $y = x^2$ $(x \geq 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

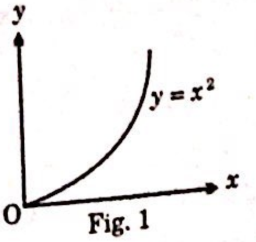


Fig. 1

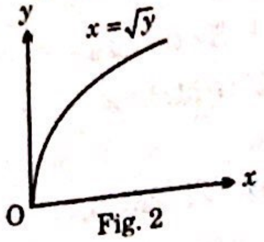


Fig. 2

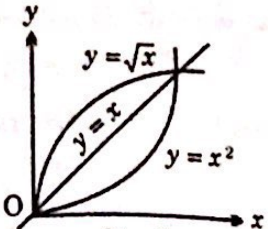


Fig. 3

(c) The inverse of a bijection is also a bijection.

(d) If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1} o g^{-1}$.

13. ODD AND EVEN FUNCTIONS :

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.

e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.

e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

Note : **(a)** $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even and $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(b) A function may neither be odd nor even.

(c) Inverse of an even function is not defined.

(d) Every even function is symmetric about the y-axis and every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even and an odd function.

e.g. $$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

(g) If f and g both are even or both are odd then the function $f.g$ will be even but if any one of them is odd then $f.g$ will be odd.

14. PERIODIC FUNCTION :

A function $f(x)$ is called periodic if there exists a positive number T $(T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of x. *e.g.* The function $\sin x$ and $\cos x$ both are periodic over 2π and $\tan x$ is periodic over π.

Note : (a) $f(T) = f(0) = f(-T)$, where 'T' is the period.

(b) Inverse of a periodic function does not exist.

(c) Every constant function is always periodic, with no fundamental period.

(d) If $f(x)$ has a period T and $g(x)$ also has a period T then it does not mean that $f(x) + g(x)$ must have a period T. *e.g.*, $f(x) = |\sin x| + |\cos x|$.

(e) If $f(x)$ has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(f) If $f(x)$ has a period T, then $f(ax + b)$ has a period T/a $(a > 0)$.

15. GENERAL :

If x, y are independent variables, then :

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(b) $f(xy) = f(x) . f(y) \Rightarrow f(x) = x^n, n \in R$

(c) $f(x + y) = f(x) . f(y) \Rightarrow f(x) = a^{kx}$.

(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

EXERCISE - 1

Only One Correct Answer

1. Let $f : R \to R$ be defined as $f(x) = 3^{-|x|} - 3^x + \text{sgn}(e^{-x}) + 2$ (where sgn x denotes signum function of x). Then which one of the following is correct?

(a) f is injective but not surjective (b) f is surjective but not injective

(c) f is injective as well as surjective (d) f is neither injective nor surjective

2. Let $f : D \to R$ be defined as $f(x) = \frac{x^2 + 2x + a}{x^2 + 4x + 3a}$ where D and R denote the domain of f and the set of all real numbers respectively. If f is surjective mapping then the range of a is :

(a) $0 \le a \le 1$ (b) $0 < a \le 1$ (c) $0 \le a < 1$ (d) $0 < a < 1$

3. If $f(x) = x^2 + bx + c$ and $f(2 + t) = f(2 - t)$ for all real numbers t, then which of the following is true?

(a) $f(1) < f(2) < f(4)$ (b) $f(2) < f(1) < f(4)$

(c) $f(2) < f(4) < f(1)$ (d) $f(4) < f(2) < f(1)$

4. If $f(x) = \pi\left(\frac{\sqrt{x+7}-4}{x-9}\right)$, then the range of function $y = \sin(2f(x))$ is :

(a) $[0, 1]$ (b) $\left(0, \frac{1}{\sqrt{2}}\right]$

(c) $\left(0, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right]$ (d) $(0, 1]$

5. Let $f(x) = \max.\{\sin t : 0 \le t \le x\}$, $g(x) = \min.\{\sin t : 0 \le t \le x\}$ and $h(x) = [f(x) - g(x)]$ where [ ] denotes greatest integer function, then the range of $h(x)$ is :

(a) $\{0, 1\}$ (b) $\{1, 2\}$

(c) $\{0, 1, 2\}$ (d) $\{-3, -2, -1, 0, 1, 2, 3\}$

6. The sum of all possible values of n where $n \in N, x > 0$ and $10 < n \le 100$ such that the equation $[2x^2] + x - n = 0$ has a solution, is equal to :

[**Note :** $[x]$ denotes largest integer less than or equal to x.]

(a) 150 (b) 175 (c) 190 (d) 210

7. If the range of the function $f(x) = \frac{x-1}{p-x^2+1}$ does not contain any values belonging to the interval $\left[-1, \frac{-1}{3}\right]$ then the true set of values of p, is :

(a) $(-\infty, -1)$ (b) $\left(-\infty, \frac{-1}{4}\right)$ (c) $(0, \infty)$ (d) $(-\infty, 0)$

8. The fundamental period of the function $f(x) = 4\cos^4\left(\frac{x-\pi}{4\pi^2}\right) - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$ is equal to :

(a) π^3 (b) $4\pi^2$ (c) $3\pi^2$ (d) $2\pi^3$

9. Let f be a function defined as $f : \left(0, e^{\frac{-3}{2}}\right] \to \left[\frac{-1}{4}, \infty\right)$, $f(x) = (\ln x)^2 + 3\ln x + 2$, then $f^{-1}(x)$ equals :

(a) $\log\left(\frac{-3+\sqrt{4x+1}}{2}\right)$ (b) $\log\left(\frac{-3-\sqrt{4x+1}}{2}\right)$

(c) $e^{\frac{-3+\sqrt{4x+1}}{2}}$ (d) $e^{\frac{-3-\sqrt{4x+1}}{2}}$

10. Let $f : X \to Y$ be defined as $f(x) = \sin x + \cos x + 2\sqrt{2}$. If f is invertible, then $X \to Y$, is :

(a) $\left[\frac{-3\pi}{4}, \frac{-\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$ (b) $\left[\frac{-\pi}{4}, \frac{3\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$

(c) $\left[\frac{-3\pi}{4}, \frac{3\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$ (d) $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to [\sqrt{2}, 3\sqrt{2}]$

11. If the range of function $f(x)=\dfrac{x^2+x+c}{x^2+2x+c}$, $x \in R$ is $\left[\dfrac{5}{6}, \dfrac{3}{2}\right]$, then c is equal to :

(a) -4 (b) 3 (c) 4 (d) 5

12. Let $f: R \to [1, \infty)$ be defined as $f(x) = \log_{10}(\sqrt{3x^2-4x+k+1}+10)$. If $f(x)$ is surjective, then :

(a) $k=\dfrac{1}{3}$ (b) $k<\dfrac{1}{3}$ (c) $k>\dfrac{1}{3}$ (d) $k=1$

13. If the domain of function $f(x)=\sqrt{\log_2\left(\dfrac{x-2}{3-x}\right)}$ is $[a, b)$, then the value of $(2a-b)$ is :

(a) 2 (b) 3 (c) 4 (d) 5

14. The range of function $f(x) = \text{sgn}(\sin x) + \text{sgn}(\cos x) + \text{sgn}(\tan x) + \text{sgn}(\cot x)$, $x \neq \dfrac{n\pi}{2}$ $(n \in I)$ is :

[**Note :** sgn k denotes signum function of k.]

(a) $\{-2, 4\}$ (b) $\{-2, 0, 4\}$ (c) $\{-4, -2, 0, 4\}$ (d) $\{0, 2, 4\}$

15. If a polynomial function 'f' satisfies the relation

$$\log_2[f(x)] = \log_2\left(2+\frac{2}{3}+\frac{2}{9}+\ldots..+\infty\right)\cdot\log_3\left(1+\frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$$ and $f(10)=1001$, then the value of $f(20)$ is :

(a) 2002 (b) 7999 (c) 8001 (d) 16001

16. If $x=\dfrac{4l}{1+l^2}$ and $y=\dfrac{2-2l^2}{1+l^2}$ where 'l' is a parameter and range of $f(x,y)=x^2-xy+y^2$ is $[a, b]$ then $(a+b)$ is equal to :

(a) 4 (b) 6 (c) 8 (d) 12

17. If the equation $|x-2|-|x+1|=p$ has exactly one solution, then number of integral values of p, is :

(a) 3 (b) 4 (c) 5 (d) 7

18. Let a, b are positive real numbers such that $a-b=10$, then the smallest value of the constant K for which $\sqrt{(x^2+ax)}-\sqrt{(x^2+bx)}<K$ for all $x>0$, is :

(a) 2 (b) 3 (c) 4 (d) 5

19. The sum of all possible solution(s) of the equation

$$\big||x+2|-3\big| = \text{sgn}\left(1-\left|\frac{(x-2)(x^2+10x+24)}{(x^2+1)(x+4)(x^2,+4x-12)}\right|\right)$$ is :

(a) 0 (b) -8 (c) -10 (d) not applicable

[**Note :** sgn (y) denotes the signum function of y.]

20. If the equation $(p^2-4)(p^2-9)x^3+\left[\frac{p-2}{2}\right]x^2+(p-4)(p^2-5p+6)x+\{2p-1\}=0$ i satisfied by all values of x in (0, 3] then sum of all possible integral values of 'p' is

(a) 0 (b) 5 (c) 9 (d) 10

[**Note :** $\{y\}$ and $[y]$ denote fractional part function and greatest integer function o y respectively.]

21. The number of integral values of x satisfying the equation $\operatorname{sgn}\left(\left[\frac{15}{1+x^2}\right]\right)=[1+\{2x\}]$ is :

(a) 5 (b) 7 (c) 15 (d) 16

[**Note :** sgn(y), [y] and {y} denote signum function, greatest integer function and fractional part function respectively.]

22. Let $f(x)=\sin^{23}x-\cos^{22}x$ and $g(x)=1+\frac{1}{2}\tan^{-1}|x|$, then the number of values of x in interval $[-10\pi, 20\pi]$ satisfying the equation $f(x)=\operatorname{sgn}(g(x))$, is :

(a) 6 (b) 10 (c) 15 (d) 20

23. The maximum value of the function $f(x)=\frac{x^4-x^2}{x^6+2x^3-1}$ where $x>1$ is equal to :

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{18}$ (d) $\frac{1}{2}$

24. The sum of all positive integral values of 'a', $a\in[1, 500]$ for which the equation $[x]^3+x-a=0$ has solution is :

[**Note :** [ ] denotes the greatest integer function.]

(a) 462 (b) 512 (c) 784 (d) 812

25. The function $f: R\to R$ defined as $f(x)=\frac{1}{2}\ln\left(\sqrt{\sqrt{x^2+1}+x}+\sqrt{\sqrt{x^2+1}-x}\right)$ is :

(a) One-one and onto both (b) One-one but not onto

(c) Onto but not one-one (d) Neither one-one nor onto

26. If $f(x)=\frac{4x(x^2+1)}{x^2+(x^2+1)^2}$, $x\ge 0$ then range of $f(x)$ is :

(a) $\left[\frac{8}{5},\infty\right)\cup\{0\}$ (b) $\left[0,\frac{8}{5}\right]$ (c) $\left(0,\frac{8}{5}\right]$ (d) $\left[2,\frac{8}{5}\right]$

27. If $f(x)=\{x\}+\left\{x+\left[\frac{x}{1+x^2}\right]\right\}+\left\{x+\left[\frac{x}{1+2x^2}\right]\right\}+\ldots\ldots+\left\{x+\left[\frac{x}{1+99x^2}\right]\right\}$ then value of $[f(\sqrt{3})]$ is :

(a) 5050 (b) 4950 (c) 17 (d) 73

Note : [k] and {k} denote greatest integer and fractional part functions of k respectively.

28. Let A = {1, 2, 3, 4}. Number of function $f : A \to A$ are three satisfying $fof\,(x) = x \,\forall\, x \in A$, is :

(a) 10 (b) 11 (c) 12 (d) 13

29. Let a function f defined from $R \to R$ as $f(x) = \begin{cases} m - x & \text{for } x \le 1 \\ 2mx + 1 & \text{for } x > 1 \end{cases}$. If the function is onto on R, then the range of m, is :

(a) $[-2, \infty)$ (b) $[-2, 0)$ (c) $\{-2\}$ (d) $(0, \infty)$

30. Let $f(x) = \begin{cases} x^2 - 3, & x \le -5 \\ x + \lambda & -5 < x < -1 \\ (\mu - 7)(|1 - x| + |1 + x|) & -1 \le x \le 1 \\ x + 6 & 1 < x < 5 \\ 3 - x^2 & x \ge 5 \end{cases}$

If $f(x)$ is an odd function then the value of $(\lambda + \mu)$, is :

(a) 1 (b) 5 (c) 10 (d) 13

31. Set A consists of 6 different elements and set B consists of 4 different elements. Number of mappings which can be defined from the set $A \to B$ which are surjective, is :

(a) 256 (b) 432 (c) 840 (d) 1560

32. If the functions $f(x) = (k^2 - 3k + 2)\,x^2 + (k^2 - 1) \,\forall\, x \in R$ and $g(x) = (k^2 - 6k + 5)\,x^3 + (k^2 - 2k + 1)\,x + (k^2 - k) \,\forall\, x \in R$ have the same graph, then the number of real values of k, is :

(a) 0 (b) 1 (c) 2 (d) 3

33. Let $f(x)$ and $g(x)$ are two functions such that $f(x) = \dfrac{2}{\pi}\tan^{-1} x$ and $g(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ then which of the following statement(s) is/are correct for the composite functions $fog : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $gof : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$?

(a) fog is a bijective function (b) fog is surjective

(c) gof is bijective (d) gof is into function

34. The function $f : [0, 2] \to [1, 4]$, defined by $f(x) = x^3 - 5x^2 + 7x + 1$, is :

(a) one-one and onto (b) onto but not one-one

(c) one-one but not onto (d) neither one-one nor onto

35. Let $f(x) = |x - 2|$ and $g(x) = \underbrace{f(f(f(f.....(f(x)))...)}_{n \text{ times}}$. If the equation $g(x) = k, k \in (0, 2)$ has 8 distinct solutions then the value n is equal to :

(a) 3 (b) 4 (c) 5 (d) 8

36. Let $f(x) = 4x(1-x)$, $0 \le x \le 1$. The number of solution of $f(f(f(x))) = \frac{x}{3}$ is :

(a) 2 (b) 4 (c) 8 (d) 16

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 and 2

Consider a quadratic function $f(x) = ax^2 + bx + c$, $(a, b, c \in R, a \ne 0)$ and satisfying the following conditions.

(a) $f(x-4) = f(2-x)\ \forall\ x \in R$ and $f(x) \ge x\ \forall\ x \in R$

(b) $f(x) \le \left(\frac{x+1}{2}\right)^2\ \forall\ x \in (0, 2)$

(c) The minimum value of $f(x)$ is zero.

1. The value of the leading coefficient of the quadratic polynomial is :

(a) 1/4 (b) 1/3 (c) 1/2 (d) 1

2. $f'(1)$ has the value equal to :

(a) 1/4 (b) 1/3 (c) 1/2 (d) 1

Paragraph for Question Nos. 3 to 5

An even periodic function $f: R \to R$ with period 4 is such that

$$f(x) = \begin{cases} \max.(|x|, x^2) & ; 0 \le x < 1 \\ x & ; 1 \le x \le 2 \end{cases}$$

3. The value of $\{f(x)\}$ at $x = 5.12$ (where { } represents fractional part), is

(a) $\{f(7.88)\}$ (b) $\{f(3.26)\}$ (c) $\{f(2.12)\}$ (d) $\{f(5.88)\}$

4. The equation of circle with centre lies on the curve $f(x)$ at $x = 9$ and touches x-axis, is :

(a) $x^2 + y^2 - 14x - 2y + 49 = 0$ (b) $x^2 + y^2 - 18x - 4y + 84 = 0$

(c) $x^2 + y^2 - 18x - 2y + 81 = 0$ (d) $x^2 + y^2 - 18x + 2y + 81 = 0$

5. If $g(x) = |3 \sin x|$, then the number of solutions of $f(x) = g(x)$ for $x \in (-6, 6)$, are :

(a) 5 (b) 7 (c) 3 (d) 9

Paragraph for Question Nos. 6 and 7

Let f be an even function satisfying

$$f(x-2) = f\left(x + \left[\frac{6x^2+13}{x^2+2}\right]\right) \ \forall\, x \in R \text{ and } f(x) = \begin{cases} 3x, & 0 \le x < 1 \\ 4-x, & 1 \le x \le 4 \end{cases}$$

[Note : [y] denotes greatest integer function of y.]

6. The area bounded by the graph of $f(x)$ and the x-axis from $x = -1$ to $x = 9$ is :

(a) $\frac{31}{2}$ (b) 15 (c) 12 (d) $\frac{15}{2}$

7. The value of $f(-89) - f(-67) + f(46)$ is equal to :

(a) 4 (b) 5 (c) 6 (d) 7

Paragraph for Question Nos. 8 to 10

$$\text{Let } f(x) = \begin{cases} \beta x^2 + 3, & -\infty < x < -1 \\ 2x + \alpha, & -1 \le x < \infty \end{cases} \text{ and } g(x) = \begin{cases} x+4, & 0 \le x \le 8 \\ -3x-2, & -\infty < x < 0 \end{cases}$$

8. The function $g(f(x))$ is not defined if :

(a) $\alpha \in (10, \infty), \beta \in (5, \infty)$ (b) $\alpha \in (4, 10), \beta \in (5, \infty)$

(c) $\alpha \in (10, \infty), \beta \in (0, 1)$ (d) $\alpha \in (4, 10), \beta \in (1, 5)$

9. If $\alpha = 2$ and $\beta = 3$, then range of $g(f(x))$ is equal to :

(a) $(-2, 12]$ (b) $(0, 12]$ (c) $[4, 12]$ (d) $[-1, 12]$

10. If $\alpha = 3$, then the value of x in interval $[1, 3]$ for which $f(x) + g(x) = 12$, is

(a) 1 (b) $\frac{3}{2}$ (c) $\frac{5}{3}$ (d) 3

Paragraph for Question Nos. 11 to 13

A line PQ parallel to the diagonal BD of a square $ABCD$ with side length 'a' unit is drawn at a distance x from the vertex A, where $x \in [0, \sqrt{2}a]$ cuts the adjacent sides at P and Q. Let $f(x)$ be the area of the segment of a square cut off by PQ, with A as one of the vertex.

11. Let $g(x) = f^{-1}(x)$, then the domain of $g(x)$ is :

(a) $x \in [0, \sqrt{2}a]$ (b) $x \in [0, 2a^2]$ (c) $x \in [\sqrt{2}a, a^2]$ (d) $x \in [0, a^2]$

12. For $a = 2$, the domain of the function $\phi(x) = \sqrt{f^{-1}(x) - f(x)}$ is/are :

(a) $x \in [0, 1]$ (b) $x \in [0, 2]$ (c) $x \in [1, 2]$ (d) $x \in [2, \infty)$

13. If the equation $f(x) = f^{-1}(x)$ has exactly three solutions $\forall\, x \in [0, \sqrt{2}a]$, then the value of a is :

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$

Paragraph for Question Nos. 14 to 16

Consider, $f(x)=\begin{cases}\sqrt{1-x^2}, & -1\le x<0\\ x^2+1, & 0\le x<1\\ \dfrac{(x-1)^2}{4}+2, & x\ge 1\end{cases}$

One more function g is defined such that $g(f(x))=x\ \forall\ x\ge -1$ and $f[g(x)]=x\ \forall\ x\ge 0$.

14. The range of the function $y=f(f(f(g(x))))$ is :
   (a) $[-1,\infty)$
   (b) $[0,\infty)$
   (c) $[1,\infty)$
   (d) $[2,\infty)$

15. The domain of $y=g(g(g(f(x))))$ is :
   (a) $[-1,\infty)$
   (b) $[0,\infty)$
   (c) $[1,\infty)$
   (d) $[2,\infty)$

16. The number of solution(s) of the equation $f(x)=g(x)$ is (are) :
   (a) 0
   (b) 1
   (c) 2
   (d) 3

Paragraph for Question Nos. 17 to 19

Consider, $f(x)=[x]^2-[x+6]$ and $g(x)=3kx^2+2x+4(1-3k)$ where $[\alpha]$ denotes the largest integer less than or equal to α.

Let $A=\{x\mid f(x)=0\}$ and $k\in[a,b]$ for which every element of set A satisfies the inequality $g(x)\ge 0$.

17. The set A is equal to :
   (a) $\{-2, 3\}$
   (b) $(-3,-2]\cup[3,4)$
   (c) $[-2,-1)\cup[3,4)$
   (d) $[-2,4]$

18. The value of $(6b-3a)$ is equal to :
   (a) 1
   (b) -1
   (c) -2
   (d) 2

19. If $k=a$ and $g:A\to B$ is onto then set B is equal to :
   (a) $[0,5)$
   (b) $(-5,5)$
   (c) $(-5,0]$
   (d) $[-5,5]$

EXERCISE - 3

More Than One Correct Answers

1. The maximum value of the function defined by $f(x) = \min(e^x, 2 + e^2 - x, 8)$ is α, then integral value of x satisfying the inequality $\frac{x(x-[a])}{x^2-[a]x+12} < 0$, is :

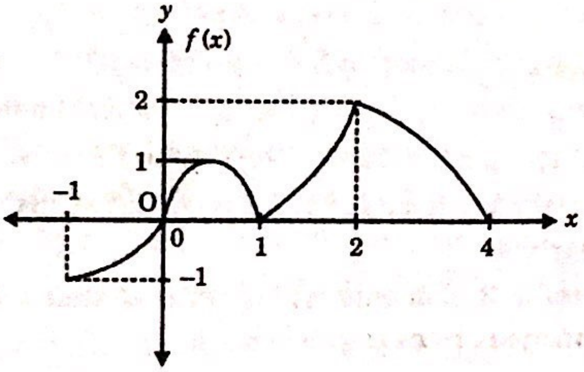
[Note : $[k]$ denotes greatest integer function less than or equal to k.]

(a) 1 (b) 3 (c) 5 (d) 6

2. If graph of a function $f(x)$ which is defined in $[-1, 4]$ is shown in the adjacent figure then identify the correct statement(s).

(a) domain of $f(|x|-1)$ is $[-5, 5]$ (b) range of $f(|x|+1)$ is $[0, 2]$

(c) range of $f(-|x|)$ is $[-1, 0]$ (d) domain of $f(|x|)$ is $[-3, 3]$



3. Let f, g and h be three functions defined as follows :

$f(x) = \frac{32}{4+x^2+x^4}$, $g(x) = 9 + x^2$ and $h(x) = -x^2 - 3x + k$.

Identify which of the following statement(s) is(are) correct?

(a) Number of integers in the range of $f(x)$ is 8.

(b) Number of integral values of k for which $h(f(x)) > 0$ and $h(g(x)) < 0\ \forall\ x \in R$ is 20.

(c) Number of integral values of k for which $h(f(x)) > 0$ and $h(g(x)) < 0 \forall\ x \in R$ is 19.

(d) Maximum value of $g(f(x))$ is 73.

4. Consider, $f(x) = \{x + [\log_2(2+x)]\} + \{x + [\log_2(2+x^2)]\} + ... + \{x + [\log_2(2+x^{10})]\}$.

Identify the correct statement(s) :

(a) $[f(e)] = 7$

(b) $f(\pi) = 20\pi - 60$

(c) the number of solutions of the equation $f(x) = x$ is 9

(d) the number of solutions of the equation $f(x) = x$ is 10

[**Note :** {y} and [y] denote the fractional part function and greatest integer function respectively.]

5. Let $f(x)=\frac{1}{x}$, $g(x)=\frac{1}{4x^2-1}$ and $h(x)=\frac{5x}{x+2}$ be three functions and $k(x)=h(g(f(x)))$. If domain and range of $k(x)$ are $R-\{a_1, a_2, a_3, ..., a_n\}$ and $R-A$ respectively where 'R' is the set of real numbers then :

(a) $n+\sum_{i=1}^{n} a_i = 5$ (b) $n+\sum_{i=1}^{n} a_i = 10$

(c) number of integers in set A is 5 (d) Number of integers in set A is 7

6. Consider, $P=\frac{x^2-2x}{x^2+x+1}$, $Q=\frac{y-1}{y^2+y+1}$ and $R=\frac{2}{z^2+z+1}$ where $x, y, z \in R$.

If $k=[P+Q+R]-([P]+[Q]+[R])$ then the possible value(s) of k is (are) :

(a) 0 (b) 1 (c) 2 (d) 3

[**Note :** [λ] denotes the greatest integer less than or equal to λ.]

7. Which of the following statement(s) is (are) correct?

(a) If f is a one-one mapping from set A to A, then f is onto.

(b) If f is an onto mapping from set A to A, then f is one-one.

(c) Let f and g be two functions defined from $R \to R$ such that gof is injective, then f must be injective.

(d) If set A contains 3 elements while set B contains 2 elements, then total number of functions from A to B is 8.

8. Which of the following are identical functions?

(a) $f(x) = \text{sgn}(|x|+1)$ (b) $g(x) = \sin^2(\ln x) + \cos^2(\ln x)$

(c) $h(x) = \frac{2}{\pi}(\sin^{-1}\{x\}+\cos^{-1}\{x\})$ (d) $k(x) = \sec^2[\{x\}]-\tan^2\{[x]\}$

[**Note :** [x] denotes greatest integer less than or equal x, {x} denotes fractional part of x and sgn x denotes signum function of x respectively.]

9. If the function $f(x)=ax+b$ has its own inverse then the ordered pair (a, b) can be:

(a) (1, 0) (b) (–1, 0) (c) (–1, 1) (d) (1, 1)

10. Let $f: A \to B$ and $g: B \to C$ be two functions and $gof: A \to C$ is defined. Then which of the following statement(s) is(are) incorrect?

(a) If gof is onto then f must be onto

(b) If f is into and g is onto then gof must be onto function

(c) If gof is one-one then g is not necessarily one-one

(d) If f is injective and g is surjective then gof must be bijective mapping

11. The function $f: R \to R$, defined as $f(x) = (x^2 - x - 21)(x^2 - x - 39)$. Which of the following is(are) correct?
    (a) Neither injective nor surjective
    (b) Minimum value of $f(x)$ is -81
    (c) $f(x) = 0$ has 4 real and distinct roots
    (d) $f(x)$ is an even function.

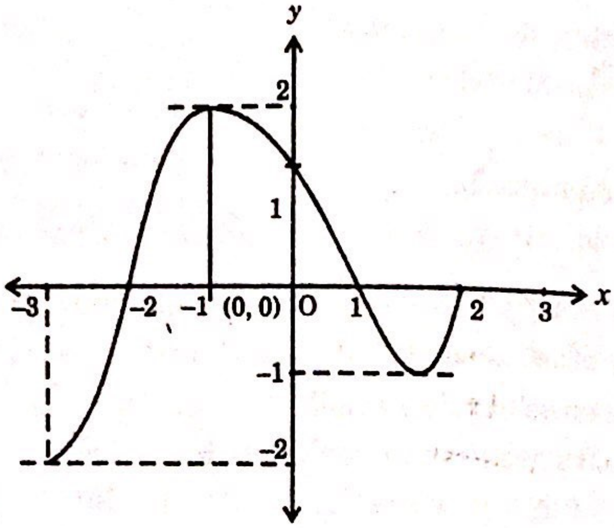
12. Let $f(x) = \left| |x^2 - 4x + 3| - 2 \right|$. Which of the following is/are correct?
    (a) $f(x) = m$ has exactly two real solutions of different sign $\forall\, m > 2$
    (b) $f(x) = m$ has exactly two real solutions $\forall\, m \in (2, \infty) \cup \{0\}$
    (c) $f(x) = m$ has no solutions $\forall\, m < 0$
    (d) $f(x) = m$ has four distinct real solutions $\forall\, m \in (0, 1)$

13. Let $f: A \to B$ be a function where set A contains 4 elements and set B contains 3 elements. Number of functions defined from $A \to B$ which are not surjective is also equal to :
    (a) number of natural solution of the equation $x + y + z = 11$.
    (b) number of ways in which 10 children can be divided into two groups one containing 2 and other containing 8 children.
    (c) number of ways in which 4 boys and 4 girls can be arranged alternately in a circle.
    (d) total number of divisors of the number 3600.

14. Let the function $f: (-\infty, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be given by $f(x) = \sin^{-1}\left(\log_3\left(\frac{x^2 - x + 1}{x^2 + x + 1}\right)\right)$, then :
    (a) $f\left(\frac{1}{x}\right) = -f(-x)$
    (b) $f(x)$ is a strictly increasing function in $(-\infty, \infty)$
    (c) $f(x)$ is a surjective function
    (d) $f(x)$ is a injective function.

15. Let $f: R \to [1, \infty)$ be a quadratic surjective function such that $f(2 + x) = f(2 - x)$ and $f(1) = 2$. If $g: (-\infty, \ln 2] \to [1, 5)$ is given by $g(\ln x) = f(x)$ then which of the following is(are) correct?
    (a) The value of $f(3)$ is equal to 2.
    (b) $g^{-1}(x) = \ln(2 - \sqrt{x - 1})$
    (c) $g^{-1}(x) = \ln(2 + \sqrt{x - 1})$
    (d) The sum of values of x satisfying the equation $f(x) = 5$ is 4.

16. The figure illustrates graph of the function $y = f(x)$ defined in $[-3, 2]$. Identify correct statement(s) :



(a) Range of $y = f(-|x|)$ is $[-2, 2]$

(b) Domain of $y = f(|x|)$ is $[-2, 2]$

(c) Domain of $y = f(|x| + 1)$ is $[-1, 1]$

(d) Range of $y = f(|x| + 1)$ is $[-1, 0]$

17. Possible integral values of x which can lie in the domain of the function : $f(x) = \log(ax^3 + (a+b)x^2 + (b+c)x + c)$ if $b^2 - 4ac < 0$ and $a > 0$, is :

(a) -1 (b) 0 (c) 1 (d) 2

18. If $a_1, a_2, a_3, \ldots, a_n$ be the integers which are not included in the range of $f(x) = \frac{40}{(x-5)(x+1)}$, where $a_i > a_{i+1}\ \forall i$ and value of $\sum_{i=1}^{n} {}^{5-a_i}C_{i-1} = {}^{x}C_{y}$, then $x +$ can be :

(a) 14 (b) 21 (c) 16 (d) 15

19. Let $f(x) = \begin{cases} x+3 & \text{if } x \in [-4, -2) \\ 1 & \text{if } x \in [-2, 2) \\ 3-x & \text{if } x \in [2, 4] \end{cases}$, $g(x) = \begin{cases} x+6 & x < 0 \\ 2x+6 & x \geq 0 \end{cases}$ then :

(a) $gof(x) = k$ will have one atleast solution if $k \in [5, 8]$

(b) Range of $fog(x)$ is $[-1, 1]$

(c) $\lim_{x \to -2} fog(x) = -1$

(d) $gof(x)$ is an even function

EXERCISE - 4

Match the Columns Type

1. Column I — **Column II**

(a) Let $f : [-1, \infty) \to (0, \infty)$ defined by $f(x) = e^{x^2+|x|}$, then $f(x)$ is — (p) one-one

(b) Let $f : (1, \infty) \to [3, \infty)$ defined by $f(x) = \sqrt{10 - 2x + x^2}$, then $f(x)$ is — (q) into

(c) Let $f : R \to I$ defined by $f(x) = \tan^5 \pi [x^2 + 2x + 3]$ where [ ] denotes greatest integer function, then $f(x)$ is — (r) many one

(d) Let $f : [3, 4] \to [4, 6]$ defined by $f(x) = |x-1| + |x-2| + |x-3| + |x-4|$ then $f(x)$ — (s) onto

(t) periodic

2. Column I — **Column II**

(a) $f(x) = \cos\left(\frac{\pi}{\sqrt{3}}\sin x + \sqrt{\frac{2}{3}}\pi\cos x\right)$ — (p) Domain of $f(x)$ is $(-\infty, \infty)$

(b) $f(x) = \log_2 (|\sin x| + 1)$ — (q) Range of $f(x)$ contains only one positive integer

(c) $f(x) = \cos \{[x] + [-x]\}$ — (r) $f(x)$ is many-one function

(d) $f(x) = [\{|e^x|\}]$ — (s) $f(x)$ is constant function

where $[x]$ and $\{x\}$ denotes greatest integer and fractional part function.

3. Column I — **Column II**

(a) Let $f(x) = \frac{1}{1-x}$. Let $f_2(x)$ denotes $f(f(x))$ and $f_3(x)$ denotes $f(f(f(x)))$. Number of distinct real in where $f_{3n}(x)$ is not defined is : — (p) 1

(b) A function $f : R \to R$ satisfies the condition, $x^2 f(x) + f(1-x) = 2x - x^4$. The value of $|f(2)|$ is : — (q) 2

(c) A function $f : \left[\frac{1}{2}, \infty\right) \to \left[\frac{3}{4}, \infty\right)$ defined as, $f(x) = x^2 - x + 1$. The value(s) of x satisfying the equation $f(x) = f^{-1}(x)$, is — (r) 3

(s) 4

4. Column I / **Column II**

(a) $f: R \to (-\infty, 0]$
$f(x) = \ln(\sqrt{1+x^4} - x^2)$ (p) many-one, into and even

(b) $g: R \to R$
$g(x) = \ln(\sqrt{1+x^6} - x^3)$ (q) many-one, onto and even

(c) $h: R \to [0, \infty)$
$h(x) = (1 + \sqrt{|x|}) + (\sqrt{|x|} + |x|)$ (r) one-one, onto and odd

(d) $k: R \to [0, 4]$
$k(x) = \dfrac{3}{x^4 - x^2 + 1}$ (s) one-one, into and odd

5. Let 'f' be a function defined in $[-2, 3]$ given as :

$$f(x) = \begin{cases} 3(x+1)^{1/3}, & -2 \le x < 0 \\ -(x-1)^2, & 0 \le x < 1 \\ 2(x-1)^2, & 1 \le x < 2 \\ -x^2 + 4x - 3, & 2 \le x \le 3 \end{cases}$$

Column I	Column II
(a) The number of integers in the range of $f(x)$ is	(p) 2
(b) The number of integral values of x which are in the domain of $f(1 - \lvert x \rvert)$, is	(q) 4
(c) The number of integers in the range of $\lvert f(-\lvert x \rvert) \rvert$, is	(r) 6
(d) The number of integral values of k for which the equation $f(\lvert x \rvert) = k$ has exactly four distinct solutions is	(s) 7

6. Column I gives functions and column II gives the nature of the functions.

Column I	Column II
(a) $f: [0, \infty) \to [0, \infty), f(x) = \dfrac{x}{1+x}$	(p) one-one onto
(b) $f: R - \{0\} \to R, f(x) = x - \dfrac{1}{x}$	(q) one-one but not onto
(c) $f: R - \{0\} \to R, f(x) = x + \dfrac{1}{x}$	(r) onto but not one-one
(d) $f: R \to R, f(x) = 2x + \sin x$	(s) neither one-one nor onto

7. Column I / **Column II**

(a) $f: R \to R, f(x) = \left[x + \dfrac{1}{2}\right] + \left[x - \dfrac{1}{2}\right] + 2[-x]$ (p) one-one

where [ ] denotes greatest integer function

(b) $f: R \to R, f(x) = x^3 + x^2 + 3x + \sin x$ (q) many on

(c) $f: R \to R, f(x) = \dfrac{e^{|x|} - e^{-x}}{e^{x} + e^{-x}}$ (r) onto

(d) $f: R \to R, f(x) = e^{\sin\{x\}} + \sin\left(\dfrac{\pi}{2}[x]\right)$ (s) into

where { } and [ ] denote fractional part function and greatest integer respectively.

EXERCISE - 5

Integer Answer Type

1. Let d be the number of integers in the range of the function
$f(x) = \begin{cases} 4, & \text{if } -4 \le x < -2 \\ |x|, & \text{if } -2 \le x < 7 \\ \sqrt{x}, & \text{if } 7 \le x < 14 \end{cases}$. Also roots of $P(x) = x^2 + mx - 4m + 20$ are α and β.
If $\alpha < \dfrac{d-3}{4} < \dfrac{d-3}{2} < \beta$ and the smallest integral value of m is k, then find the value of $(k - 5)$.

2. If $f: [2, \infty) \to [8, \infty)$ is a surjective function defined by $f(x) = x^2 - (p-2)x + 3p - 2$, $p \in R$ then sum of values of p is $m + \sqrt{n}$, where $m, n \in$ N. Find the value of $\dfrac{n}{m}$.

3. Let $f(x) = -x^{100}$. If $f(x)$ is divided by $x^2 + x$, then the remainder is $r(x)$. Find the value of $r(10)$.

4. $f: R \to R, f(x) = \dfrac{3x^2 + mx + n}{x^2 + 1}$. If the range of this function is $[-4, 3)$, then find the value of $m^2 + n^2$.

5. Let $f(x) = ax^2 + bx + c$ $(a < b)$ and $f(x) \ge 0\ \forall\, x \in R$. Find the minimum value of $\dfrac{a+b+c}{b-a}$.

6. Let $f(x) = |x^2 - 9| - |x - a|$. Find the number of integers in the range of a so that $f(x) = 0$ has 4 distinct real root.

7. If $f(x) = x^3 + 3x^2 + 4x + b \sin x + c \cos x\ \forall\, x \in R$ is a one-one function, then find the maximum value of $(b^2 + c^2)$.

8. Let the range of the function $f: \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ assumes exactly 3 distinct values. If the number of such function is N, then find the value of $\dfrac{N}{300}$.

9. The set of real values of 'x' satisfying the equality $\left[\frac{3}{x}\right]+\left[\frac{4}{x}\right]=5$ (where [ ] denotes the greatest integer function) belongs to the interval $\left(a, \frac{b}{c}\right]$ where $a, b, c \in N$ and $\frac{b}{c}$ is in its lowest form. Find the value of $a + b + c + abc$.

10. Let $[x]$ = the greatest integer less than or equal to x. If all the values of x such that the product $\left[x-\frac{1}{2}\right]\left[x+\frac{1}{2}\right]$ is prime, belongs to the set $[x_1, x_2) \cup [x_3, x_4)$, find the value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$.

11. Let $f(x)=\left[\frac{1}{\cos\{x\}}\right]$ where $[y]$ and $\{y\}$ denote greatest integer and fractional part functions respectively and $g(x) = 2x^2 - 3x(k+1) + k(3k+1)$. If $g(f(x)) < 0 \ \forall x \in R$ then find the number of integral values of k.

12. The polynomial $R(x)$ is the remainder upon dividing x^{2007} by $x^2 - 5x + 6$. If $R(0)$ can be expressed as $ab(a^c - b^c)$, find the value of $(a + b + c)$.

13. Let function $f(x)$ be defined as $f(x) = x^2 + bx + c$, where b, c are real numbers and $f(1) - 2f(5) + f(9) = 32$. Number of ordered pairs (b, c) such that $|f(x)| \le 8$ for all x in the interval $[1, 9]$.

14. Find the largest integral value of 'a' for which every solution of the equation $x([x]-5)+2\{x\}+6=0$ satisfies the inequality $(a-3)x^2+2(a+3)x-8a \le 0$, where $[y]$ and $\{y\}$ denote greatest integer and fractional part functions respectively.

15. Let $f(x)=\sqrt{\sin^4 x+4\cos^2 x}-\sqrt{\cos^4 x+4\sin^2 x}$ and $g(\sin 2t) = \sin t + \cos t \ \forall t \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$. If the range of $g(f(x))$ is $[a, b]$ then find the value of $a^2 + b^2$.

16. Find the number of integers in the range of the function

$$f(x)=\cos x\left(\sin x+\sqrt{\sin^2 x+\frac{1}{2}}\right).$$

17. Let f be an odd periodic function from R to R. If the period of f is 2 and it is given that $f\left(3.65+\log_{\frac{9}{4}}\left(\frac{1}{2\sqrt{3}}\sqrt{6-\frac{1}{2\sqrt{3}}\sqrt{6-\frac{1}{2\sqrt{3}}\ \ldots\ldots \infty}}\right)\right)=1$, then find the absolute value of $f(0.85)$.

18. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$ are two sets. Let m is the number of one-one functions $f : A \to B$ such that $f(i) \ne i \ \forall i \in A$ and n is the number of one-one functions $f : A \to B$ such that $|f(i) - i| \le 3 \ \forall i \in A$, then find the value of $(m + n)$.

19. Find the sum of all integral values of a where $a \in [-10, 10]$ such that the graph of the function $f(x) = ||x-2|-a|-3$ has exactly three x-intercepts.

20. Find the value of expression $\left[\frac{18}{35}\right]+\left[\frac{18(2)}{35}\right]+\left[\frac{18(3)}{35}\right]+\ldots\ldots+\left[\frac{18(33)}{35}\right]+\left[\frac{18(34)}{35}\right]$

[Note : $[y]$ denotes the greatest integer function less than or equal to y.]

21. Find the number of functions that can be defined from the set $A = \{1, 2, 3\}$ to the set $B = \{1, 2, 3, 4, 5\}$, such that $f(i) \le f(j)$ for $i < j$.

22. $f: R \to R, f(x) = \frac{3x^2 + mx + n}{x^2 + 1}$. If the range of this function is $[-4, 3)$, then find the value of $m^2 + n^2$.

23. Let a function $f: R \to R$ such that $f(1) = 2$ and $f(x+y) = 2^x f(y) + 4^y f(x)\ \forall\, x, y \in R$. If $f'(2) = k \ln 2$, then find the value of k.

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (d)	2. (d)	3. (b)	4. (c)	5. (c)	6. (c)	7. (b)	8. (d)	9. (d)	10. (d)
11. (c)	12. (a)	13. (a)	14. (b)	15. (c)	16. (c)	17. (c)	18. (d)	19. (d)	20. (b)
21. (b)	22. (c)	23. (b)	24. (d)	25. (d)	26. (b)	27. (d)	28. (d)	29. (b)	30. (a)
31. (d)	32. (b)	33. (d)	34. (b)	35. (b)	36. (c)				

EXERCISE 2 : Linked Comprehension Type

1. (a)	2. (d)	3. (a)	4. (c)	5. (b)	6. (b)	7. (a)	8. (a)	9. (c)	10. (c)
11. (d)	12. (a)	13. (b)	14. (d)	15. (c)	16. (d)	17. (c)	18. (d)	19. (a)	

EXERCISE 3 : More Than One Correct Answers

1. (a, c, d)	2. (a, b, c)	3. (a, c, d)	4. (a, c)	5. (a, d)
6. (a, b, c)	7. (c, d)	8. (a, c, d)	9. (a, b, c)	10. (a, b, d)
11. (a, b, c)	12. (a, b, c, d)	13. (a, b, d)	14. (a, c)	15. (a, b, d)
16. (a, b, c, d)	17. (b, c, d)	18. (a, c)	19. (a, b, c, d)	

EXERCISE 4 : Match the Columns Type

1. (a) (q) (r), (b) (p) (q), (c) (q) (r) (t), (d) (p) (s)
2. (a) (p) (q) (r), (b) (p) (q) (r), (c) (p) (q) (r) (s), (d) (p) (r) (s)
3. (a) (q), (b) (r), (c) (p)
4. (a) (q), (b) (r), (c) (p), (d) (p)
5. (a) (r), (b) (s), (c) (q), (d) (p)
6. (a) (q), (b) (r), (c) (s), (d) (p)
7. (a) (q) (s), (b) (p) (r), (c) (q) (s), (d) (q) (s)

EXERCISE 5 : Integer Answer Type

1. 8	2. 2	3. 10	4. 16	5. 3
6. 17	7. 1	8. 5	9. 20	10. 11
11. 1	12. 2011	13. 1	14. 1	15. 2
16. 3	17. 1	18. 94	19. 3	20. 289
21. 35	22. 16	23. 28		

□□□

Chapter 2

Inverse Trigonometric Functions

KEY CONCEPTS

1. GENERAL DEFINITION(S) :

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. denote angles or real numbers whose sine is x, whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available. These are also written as arc sinx, arc cosx etc.

If there are two angles one positive and the other negative having same numerical value, then positive angle should be taken.

2. PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :

(a) $y = \sin^{-1}x$ where $-1 \le x \le 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $\sin y = x$.

(b) $y = \cos^{-1}x$ where $-1 \le x \le 1$; $0 \le y \le \pi$ and $\cos y = x$.

(c) $y = \tan^{-1}x$ where $x \in R$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan y = x$.

(d) $y = \text{cosec}^{-1}x$ where $x \le -1$ or $x \ge 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$ and $\text{cosec}\, y = x$.

(e) $y = \sec^{-1}x$ where $x \le -1$ or $x \ge 1$; $0 \le y \le \pi$; $y \ne \frac{\pi}{2}$ and $\sec y = x$.

(f) $y = \cot^{-1}x$ where $x \in R$, $0 < y < \pi$ and $\cot y = x$.

Note :

(a) 1st quadrant is common to all the inverse functions.

(b) 3rd quadrant is not used in inverse functions.

(c) 4th quadrant is used in the clockwise direction *i.e.* $-\frac{\pi}{2} \le y \le 0$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

P–1 (a) $\sin(\sin^{-1}x) = x, -1 \le x \le 1$

(b) $\cos(\cos^{-1}x) = x, -1 \le x \le 1$

(c) $\tan(\tan^{-1}x) = x, x \in R$

(d) $\sin^{-1}(\sin x) = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(e) $\cos^{-1}(\cos x) = x; 0 \le x \le \pi$

(f) $\tan^{-1}(\tan x) = x; -\frac{\pi}{2} < x < \frac{\pi}{2}$

P–2 (a) $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}, x \le -1, x \ge 1$

(b) $\sec^{-1}x = \cos^{-1}\frac{1}{x}, x \le -1, x \ge 1$

(c) $\cot^{-1}x = \tan^{-1}\frac{1}{x}, x > 0$

$= \pi + \tan^{-1}\frac{1}{x}, x < 0$

P–3 (a) $\sin^{-1}(-x) = -\sin^{-1}x, -1 \le x \le 1$

(b) $\tan^{-1}(-x) = -\tan^{-1}x, x \in R$

(c) $\cos^{-1}(-x) = \pi - \cos^{-1}x, -1 \le x \le 1$

(d) $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

P–4 (a) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, -1 \le x \le 1$

(b) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in R$

(c) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, |x| \ge 1$

P–5 $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, where $x > 0, y > 0$ and $xy < 1$

$= \pi + \tan^{-1}\frac{x+y}{1-xy}$, where $x > 0, y > 0$ and $xy > 1$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$, where $x > 0, y > 0$

P–6 (a) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

where $x \ge 0, y \ge 0$ and $(x^2 + y^2) \le 1$

Note : $x^2 + y^2 \le 1 \Rightarrow 0 \le \sin^{-1}x + \sin^{-1}y \le \frac{\pi}{2}$

(b) $\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

where $x \geq 0, y \geq 0$ and $x^2 + y^2 > 1$

Note : $x^2 + y^2 > 1 \Rightarrow \frac{\pi}{2} < \sin^{-1}x + \sin^{-1}y < \pi$

(c) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$, where $x \geq 0, y \geq 0$

(d) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}]$, where $x \geq 0, y \geq 0$

P–7 If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

if, $x > 0, y > 0, z > 0$ and $xy + yz + zx < 1$

Note : (a) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

(b) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

P–8 $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$

Note :

$$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \leq 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$$

$$\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \geq 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$

$$\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$$

Remember :

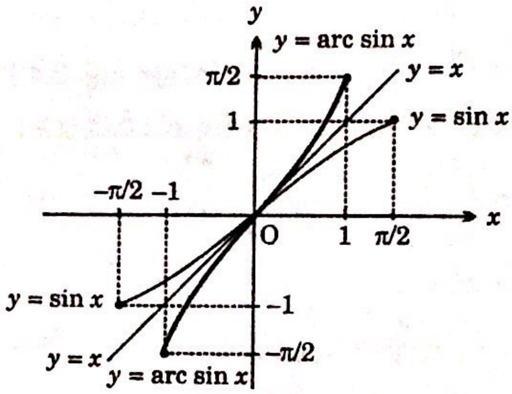
(a) $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2} \quad \Rightarrow \quad x = y = z = 1$

(b) $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi \quad \Rightarrow \quad x = y = z = -1$

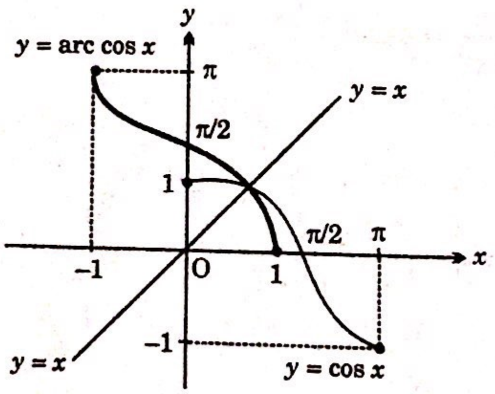
(c) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ and $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

SOME USEFUL GRAPHS

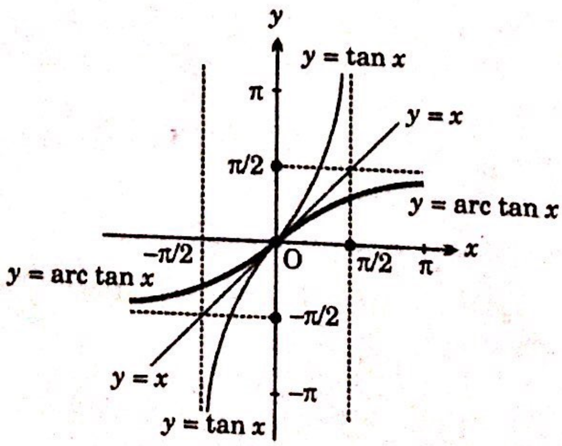
1. $y = \sin^{-1} x,\ |x| \le 1,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



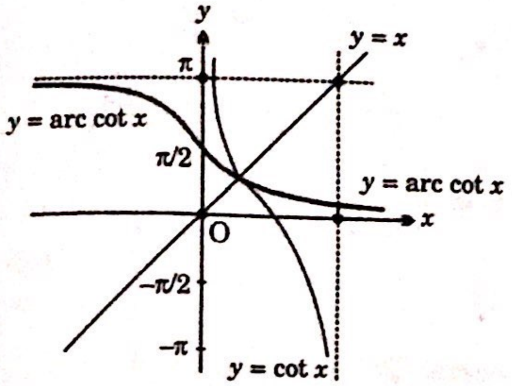
2. $y = \cos^{-1} x,\ |x| \le 1,\ y \in [0, \pi]$



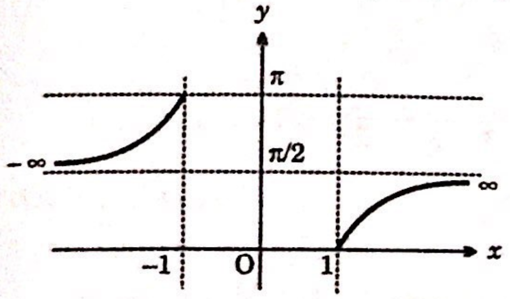
3. $y = \tan^{-1} x,\ x \in R,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



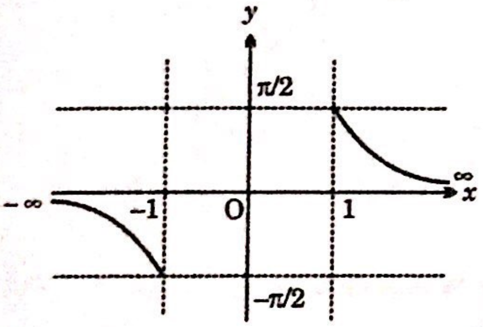
4. $y = \cot^{-1} x, x \in R, y \in (0, \pi)$



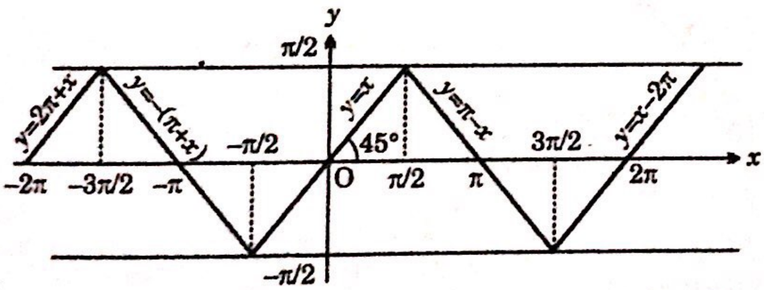
5. $y = \sec^{-1} x,\ |x| \geq 1, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



6. $y = \text{cosec}^{-1} x,\ |x| \geq 1, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



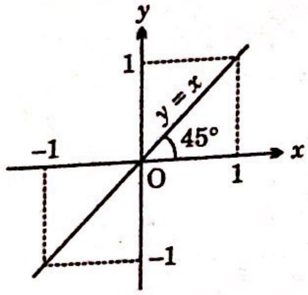
7. (a) $y = \sin^{-1}(\sin x), x \in R,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, periodic with period 2π



(b) $y = \sin(\sin^{-1} x)$,

$= x$

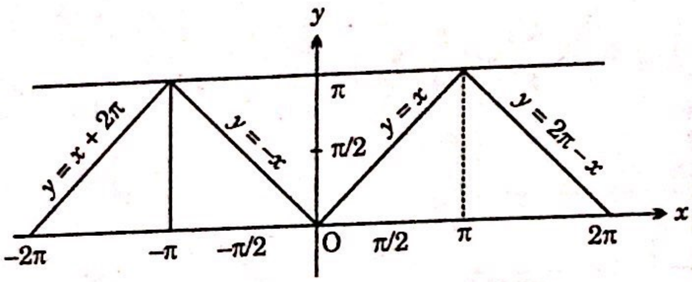
$x \in [-1, 1], y \in [-1, 1]$, y is a periodic



8. (a) $y = \cos^{-1}(\cos x)$,

$= x$

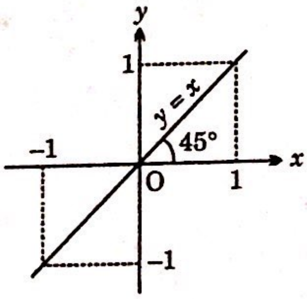
$x \in R, y \in [0, \pi]$, periodic with period 2π



(b) $y = \cos(\cos^{-1} x)$,

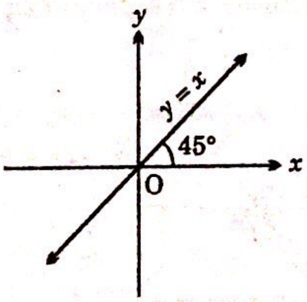
$= x$

$x \in [-1, 1], y \in [-1, 1]$, y is a periodic



9. (a) $y = \tan(\tan^{-1} x), x \in R, y \in R$, y is a periodic

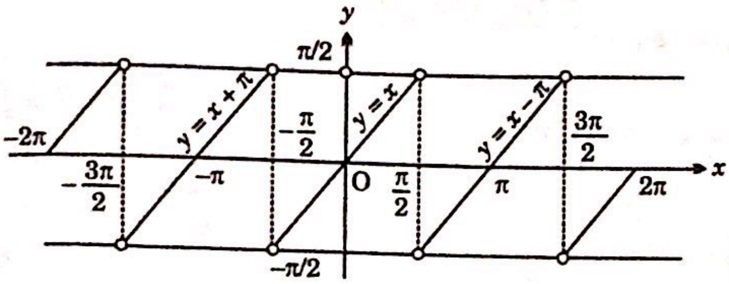
$= x$



(b) $y = \tan^{-1}(\tan x)$,

$= x$

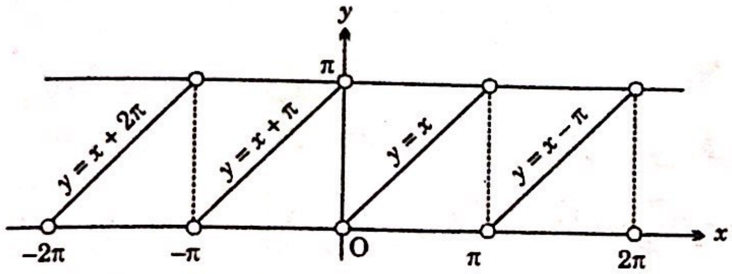
$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, periodic with period π



10. (a) $y = \cot^{-1}(\cot x)$,

$= x$

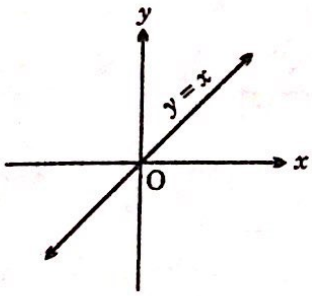
$x \in R - \{n\pi\}, y \in (0, \pi)$, periodic with π



(b) $y = \cot(\cot^{-1} x)$,

$= x$

$x \in R, y \in R$, y is a periodic

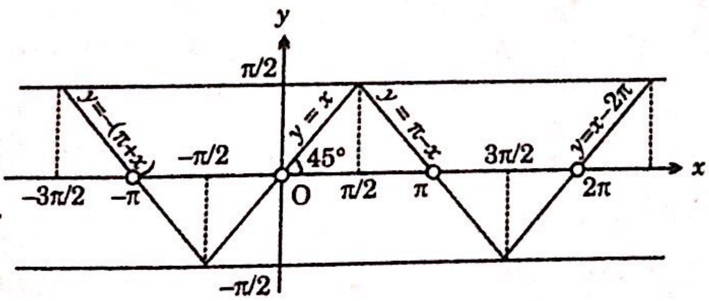


11. (a) $y = \text{cosec}^{-1}(\text{cosec}\, x)$,

$= x$

$x \in R - \{n\pi, n \in I\},\ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

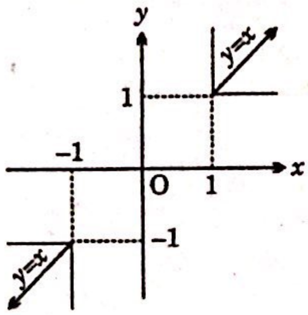
y is periodic with period 2π



(b) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x)$,

$= x$

$|x| \geq 1$, $|y| \geq 1$, y is aperiodic

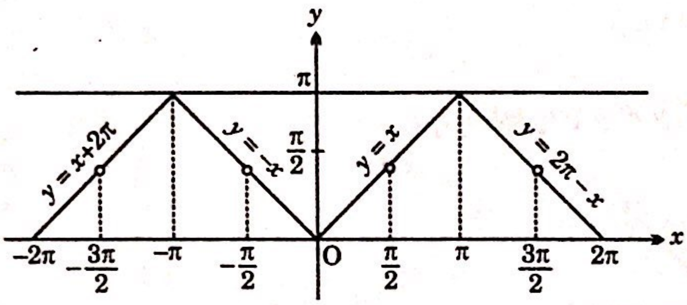


12. (a) $y = \sec^{-1}(\sec x)$,

$= x$

y is periodic with period 2π;

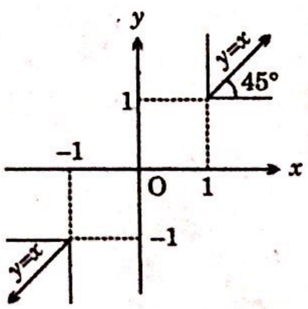
$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



(b) $y = \sec(\sec^{-1} x)$,

$= x$

$|x| \geq 1$; $|y| \geq 1]$, y is aperiodic



EXERCISE - 1

Only One Correct Answer

1. The value of $\tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\frac{\sqrt{5-2\sqrt{6}}}{1+\sqrt{6}}$ is equal :

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) none of these

2. Number of solutions of the equation $\log_{10}(\sqrt{5\cos^{-1}x-1}) + \frac{1}{2}\log_{10}(2\cos^{-1}x+3) + \log_{10}\sqrt{5} = 1$ is :

(a) 0 (b) 1
(c) more than one but finite (d) infinite

3. The number of solutions of the equation $|y| = \cos x$ and $y = \cot^{-1}(\cot x)$ in $\left(-\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ is :

(a) 2 (b) 4 (c) 6 (d) none of these

4. Equation of the image of the line $x+y=\sin^{-1}(a^6+1)+\cos^{-1}(a^4+1)-\tan^{-1}(a^2+1)$, $a \in R$ about x axis is given by :

(a) $x-y=0$ (b) $x-y=\frac{\pi}{2}$ (c) $x-y=\pi$ (d) $x-y=\frac{\pi}{4}$

5. If $\sin^{-1}\left(x^2-\frac{x^4}{3}+\frac{x^6}{9}-......\right)+\cos^{-1}\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}-......\right)=\frac{\pi}{2}$, where $0 \le |x| < \sqrt{3}$, then number of values of 'x' is equal to :

(a) 1 (b) 2 (c) 3 (d) 4

6. If $a\sin^{-1}x - b\cos^{-1}x = c$, then the value of $a\sin^{-1}x + b\cos^{-1}x$ (whenever exists) is equal to :

(a) 0 (b) $\frac{\pi ab + c(b-a)}{a+b}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi ab + c(a-b)}{a+b}$

7. The range of $f(x) = \cot^{-1}(-x) - \tan^{-1}x + \sec^{-1}x$ is :

(a) $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ (b) $\left[\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right]$
(c) $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (d) $\left(\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

8. Let $f: R \to \left(0, \frac{\pi}{6}\right]$ be defined as $f(x) = \sin^{-1}\left(\frac{4}{4x^2-12x+17}\right)$ then $f(x)$ is :

(a) injective as well as surjective (b) surjective but not injective
(c) injective but not surjective (d) neither injective nor surjective

9. The value of $\sin^{-1}(\cos 2) - \cos^{-1}(\sin 2) + \tan^{-1}(\cot 4) - \cot^{-1}(\tan 4) + \sec^{-1}(\text{cosec}$ $- \text{cosec}^{-1}(\sec 6)$ is :

(a) 0 (b) 3π (c) $8 - 3\pi$ (d) $5\pi - 16$

10. Let $g : R \to \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$ is defined by $g(x) = \sin^{-1}\left(\frac{x^2 - c}{1 + x^2}\right)$. Then the possible values of c for which g is surjective function, is :

(a) $\left\{\frac{1}{2}\right\}$ (b) $\left(-1, -\frac{1}{2}\right]$ (c) $\left\{-\frac{1}{2}\right\}$ (d) $\left[-\frac{1}{2}, 1\right)$

11. If $f(x) = \tan^{-1}x - \frac{2}{\pi}(\tan^{-1}x)^2 + \frac{4}{\pi^2}(\tan^{-1}x)^3 - \ldots\ldots\ \infty$, then the sum of integral values of a for which the equation $f^2(x) + (\sin^{-1}x)^2 = a$, possess solution, is :

(a) 6 (b) 7 (c) 9 (d) 10

12. Let $f : R \to \left(-\frac{\pi}{2}, 0\right]$ be a function defined by $f(x) = \tan^{-1}(2x - x^2 + \lambda)$. If f is onto, then λ lies in the interval :

(a) (–2, 0) (b) (0, 2) (c) (–1, 1) (d) none of these

13. The value of $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \cot^{-1}\left(2^{r+1} + \frac{1}{2^r}\right)$ is equal to :

(a) $\tan^{-1} 2$ (b) $\cot^{-1} 2$ (c) $\sec^{-1} 2$ (d) $\text{cosec}^{-1} 2$

14. If the equation $5 \text{ arc tan }(x^2 + x + k) + 3 \text{ arc cot }(x^2 + x + k) = 2\pi$, has two distinct solutions, then the range of k, is :

(a) $\left(0, \frac{5}{4}\right]$ (b) $\left(-\infty, \frac{5}{4}\right)$ (d) $\left(\frac{5}{4}, \infty\right)$ (d) $\left(-\infty, \frac{5}{4}\right]$

15. Number of solutions of the equation $|\sin^{-1}(\sin x)| = \cos x$, for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is equal to:

(a) 0 (b) 2 (c) 4 (d) 6

16. If $(\sin^{-1}x)^3 + (\sin^{-1}y)^3 + (\sin^{-1}z)^3 = \frac{(3\pi)^3}{8}$, then the value of $(2x - 3y + 4z)$ is equal to :

(a) 2 (b) 3 (c) 4 (d) 9

17. Let $S_n = \sum\limits_{r=0}^{n-1} \cos^{-1}\left(\frac{n^2 + r^2 + r}{\sqrt{n^4 + r^4 + 2r^3 + 2n^2r^2 + 2n^2r + n^2 + r^2}}\right)$, then the value of S_{100}, is :

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

18. Let $f:(-\infty,-1]\to\left(\frac{\pi}{2},\pi\right]$ be defined as $f(x)=\sec^{-1}(-x^2+x+a)$. If $f(x)$ is surjective, then the range of a is :

(a) $\{1\}$ (b) $\left\{\frac{-5}{4}\right\}$ (c) $\left(-\infty,\frac{-5}{4}\right]$ (d) $(-\infty, 1]$

19. Let $f(x)=\sin^{-1}x+\cos^{-1}x+\pi(|x|-2)$. The true set of values of k for which the equation $f(x)=k\pi$ possesses real solution is $[a, b]$ then the value of $2\ |a|+4\ |b|$ is equal to :

(a) 2 (b) 3 (c) 5 (d) 7

20. For $x\in(0, 1)$, let $\alpha=\sin^{-1}x$, $\beta=x$, $\gamma=\tan^{-1}x$, $\delta=\cot^{-1}x-\frac{\pi}{2}$, then identify the correct relation :

(a) $\alpha>\beta>\delta>\gamma$ (b) $\beta>\alpha>\gamma>\delta$ (c) $\alpha>\beta>\gamma>\delta$ (d) $\beta>\alpha>\delta>\gamma$

21. If $f(x)=\left(\left[\{x\}\right]\tan^{-1}\left(\frac{x^2-3x-1}{x^2-3x+5}\right)+3-x^7\right)^{\frac{1}{7}}$. where $[k]$ and $\{k\}$ denotes greatest integer and fractional part functions of k respectively, then the value of $f^{-1}(50)-f(50)+f(f(100))$, is :

(a) 0 (b) 25 (c) 50 (d) 100

22. The sum of series $\cot^{-1}\left(\frac{9}{2}\right)+\cot^{-1}\left(\frac{33}{4}\right)+\cot^{-1}\left(\frac{129}{8}\right)+\ldots\ldots\infty$ is equal to :

(a) $\cot^{-1}(2)$ (b) $\cot^{-1}(3)$ (c) $\cot^{-1}(-1)$ (d) $\cot^{-1}(1)$

23. The number of solution of the equation $2\sin^{-1}\left(\frac{2x}{1+x^2}\right)-\pi x^3=0$ is equal to :

(a) 0 (b) 1 (c) 2 (d) 3

24. If x_1, x_2 and x_3 are the positive roots of the equation $x^3-6x^2+3px-2p=0, p\in R$ then the value of $\sin^{-1}\left(\frac{1}{x_1}+\frac{1}{x_2}\right)+\cos^{-1}\left(\frac{1}{x_2}+\frac{1}{x_3}\right)-\tan^{-1}\left(\frac{1}{x_3}+\frac{1}{x_1}\right)$ is equal to :

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) π

25. If range of $f(x)=\cos^{-1}(x^2\cdot\cos 1+\sin 1\cdot\sqrt{1-x^4})$ is $[a, b]$, then $(b-a)$ is equal to :

(a) $\frac{\pi}{2}$ (b) π (c) $6-\frac{3\pi}{2}$ (d) 1

26. The value of $\tan\left(\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{4}{4r^2+3}\right)\right)$ is equal to :

(a) 1 (b) 2 (c) 3 (d) 4

27. Number of integral values of k for which the equation $4\cos^{-1}(-|x|)=k$ has exactly two solutions, is :

(a) 4 (b) 5 (c) 6 (d) 7

28. The exhaustive set of values of 'a' for which the function $f(x) = \tan^{-1}(x^2 - 18x + a) > 0 \ \forall\, x \in R$ is :

(a) $(81, \infty)$ (b) $[81, \infty)$ (c) $(-\infty, 81]$ (d) $(-\infty, 81)$

29. The value of sum $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{8n}{n^4 - 2n^2 + 5}\right)$ equals :

(a) $\tan^{-1}(2)$ (b) $\cot^{-1}(2)$

(c) $\cot^{-1}(2) + \cot^{-1}(3)$ (d) $\tan^{-1}(2) + \cot^{-1}(2)$

30. If 'α' is the only real root of the equation $x^3 + bx^2 + cx + 1 = 0$ $(b < c)$, then the value of $\tan^{-1}\alpha + \tan^{-1}(\alpha^{-1})$ is equal to :

(a) $\frac{-\pi}{2}$ (b) $\frac{\pi}{2}$

(c) 0 (d) can't be determined

31. The number of non zero roots of the equation $\sqrt{\sin x} = \cos^{-1}(\cos x)$ in $(0, 2\pi)$:

(a) 0 (b) 1 (c) 2 (d) infinite

32. Let $f(x) = \sin x + \cos x + \tan x + \text{arc}\sin x + \text{arc}\cos x + \text{arc}\tan x$. If M and m are maximum and minimum values of $f(x)$ then their arithmetic mean is equal to :

(a) $\frac{\pi}{2} + \cos 1$ (b) $\frac{\pi}{2} + \sin 1$

(c) $\frac{\pi}{4} + \tan 1 + \cos 1$ (d) $\frac{\pi}{4} + \tan 1 + \sin 1$

33. For $n \in N$, if $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$, then n is equal to:

(a) 43 (b) 47 (c) 49 (d) 51

34. Let $S_n = \cot^{-1}\left(3x + \frac{2}{x}\right) + \cot^{-1}\left(6x + \frac{2}{x}\right) + \cot^{-1}\left(10x + \frac{2}{x}\right) + \ldots\ldots\ldots\ldots + n$ terms, where $x > 0$. If $\lim_{n\to\infty} S_n = 1$, then x equals :

(a) $\frac{\pi}{4}$ (b) 1 (c) $\tan 1$ (d) $\cot 1$

35. Let $f : R \to [-1, 1]$ and $g : R \to B$, where R be the set of all real numbers and $g(x) = \sin^{-1}\left(\frac{f(x)}{2}\sqrt{4 - f^2(x)}\right) + \frac{\pi}{3}$. If $y = f(x)$ and $y = g(x)$ are both surjective, then set B is given by :

(a) $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ (b) $\left[0, \frac{2\pi}{3}\right]$ (c) $\left[0, \frac{\pi}{3}\right]$ (d) $[0, \pi]$

36. Let, $f(x) = \begin{cases} \frac{8}{\pi}\tan^{-1}(-|x|+3), & |x| > 2 \\ \left[\frac{3x^2 - |x| + 3}{x^2 + 1}\right], & |x| \le 2 \end{cases}$

Number of integers in the range of $f(x)$ is
where [ ] denotes greatest integer function.

(a) 5 (b) 6 (c) 7 (d) more than 7

37. If the sum of the series $\sum_{n=1}^{\infty}\left(\frac{\pi - \sec^{-1}\sqrt{|x|+1} - \operatorname{cosec}^{-1}\sqrt{|x|+1}}{\pi a}\right)^n$ is finite, where $x \in R$ and $a > 0$ then a lies in the interval :

(a) $(1, \infty)$ (b) $\left(0, \frac{1}{2}\right)$ (c) $\left(\frac{1}{2}, \infty\right)$ (d) $\left[\frac{1}{2}, \infty\right)$

38. If k times the sum of first n natural numbers is equal to the sum of squares of first n natural numbers, then $\sin^{-1}\left(\frac{9k^2 - 4n^2}{6k + 4n}\right)$ is equal to :

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

39. If $\sin^{-1}(e^x) + \cos^{-1}(x^2) = \frac{\pi}{2}$, then find the number of solutions of this equation :

(a) 1 (b) 2 (c) 3 (d) 0

40. If $f(x) = \tan^{-1}\left(\sqrt{\left(\frac{1-\cos 2}{1+\cos 2}\right)} + x^2\right)$ and $g(x) = \begin{bmatrix} |x| & x \neq 0 \\ 1 & x = 0 \end{bmatrix}$ then number of solution(s) of the equation $f(x) = g(x)$ is (are) :

(a) 0 (b) 1 (c) 2 (d) 3

41. If $f(x) = \sin^{-1}\left(\frac{4x}{4+x^2}\right)$ then the value of $f(\sqrt{7}-2) - f\left(\frac{4}{\sqrt{7}-2}\right)$ is :

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{4}$

42. The maximum value of the function $f(x) = (\sin^{-1}(\sin x))^2 - \sin^{-1}(\sin x)$ is :

(a) $\frac{\pi}{4}(\pi + 2)$ (b) $\frac{\pi}{4}(\pi - 2)$ (c) $\frac{\pi}{2}(\pi + 2)$ (d) $\frac{\pi}{2}(\pi - 2)$

43. If $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{3x}\right) + \tan^{-1}\left(\frac{3}{4}\right) = \frac{\pi}{2}$, then the value of $\tan(\pi - 2\tan^{-1} x)$ is :

(a) $\frac{24}{7}$ (b) $\frac{7}{24}$ (c) $\frac{25}{7}$ (d) $\frac{7}{25}$

44. The complete solution set of the equation

$$\sin^{-1}\sqrt{\frac{1+x}{2}} - \sqrt{2-x} = \cot^{-1}(\tan\sqrt{2-x}) - \sin^{-1}\sqrt{\frac{1-x}{2}}$$ is :

(a) $[-1, 1]$ (b) $\left(2 - \frac{\pi^2}{4}, 1\right)$ (c) $\left[-1, 2 - \frac{\pi^2}{4}\right)$ (d) $[0, 1]$

45. If the equation $x^3 + bx^2 + cx + 1 = 0$, $(b < c)$, has only one root α then the value of $2\tan^{-1}(\text{cosec}\,\alpha) + \tan^{-1}(2\sin\alpha\sec^2\alpha)$ is :

(a) $-\pi$ (b) $-\frac{\pi}{2}$ (c) $\frac{\pi}{2}$ (d) π

46. Let m be the number of elements in the domain of $f(x)$ and n be the number of elements in the range of $f(x)$ where $f(x) = \sin^{-1}[\sec(3\tan^{-1}x)] + \cos^{-1}[\text{cosec}(3\cot^{-1}x)]$, then the value of $(m + n)$ is :

(a) 3 (b) 4 (c) 5 (d) 6

47. If the minimum value of function $f(x) = 8^{\sin^{-1}x} + 8^{\cos^{-1}x}$ is m, then the value of $\log_2 m$ is equal to :

(a) $1+\frac{\pi}{4}$ (b) $-1+\frac{3\pi}{4}$ (c) $1+\frac{3\pi}{4}$ (d) $-1+\frac{\pi}{2}$

48. The value of x satisfying the equation $(\sin^{-1}x)^3 - (\cos^{-1}x)^3 + (\sin^{-1}x)(\cos^{-1}x)(\sin^{-1}x - \cos^{-1}x) = \frac{\pi^3}{16}$ is :

(a) $\cos\frac{\pi}{5}$ (b) $\cos\frac{\pi}{4}$ (c) $\cos\frac{\pi}{8}$ (d) $\cos\frac{\pi}{12}$

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 to 3

Let f be a real-valued function defined on R (the set of real numbers) such that $f(x) = \sin^{-1}(\sin x) + \cos^{-1}(\cos x)$.

1. The value of $f(10)$ is equal to :

(a) $6\pi - 20$ (b) $7\pi - 20$ (c) $20 - 7\pi$ (d) $20 - 6\pi$

2. The area bounded by curve $y = f(x)$ and x-axis from $\frac{\pi}{2} \le x \le \pi$ is equal to :

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$ (c) π^2 (d) $\frac{\pi^2}{8}$

3. Number of values of x in interval (0, 3) so that $f(x)$ is an integer, is equal to:

(a) 1 (b) 2 (c) 3 (d) 0

Paragraph for Question Nos. 4 and 5

Let f be a monic biquadratic polynomial satisfying $f(-x) = f(x)$ for all $x \in R$ and having minimum value -4 at $x = \pm 2$.

4. The number of integral values of k for which the equation $f(x) = k$ has four distinct real solutions, is :

(a) 2 (b) 7 (c) 15 (d) 21

5. The value of $\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{8r}{f(r)+5}\right)$ is equal to :

(a) $2\pi-\tan^{-1}4$ (b) $\pi-\tan^{-1}4$ (c) $\frac{3\pi}{2}-\tan^{-1}4$ (d) $\frac{\pi}{2}-\tan^{-1}4$

Paragraph for Question Nos. 6 to 8

Let $\alpha = 2\tan^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}$ and $\beta=\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\cot^{-1}\frac{16}{63}$ be such that $2\sin\alpha$ and $\cos\beta$ are roots of the equation $x^2-ax+b=0$, where $a, b \in R$.

6. The value of $\tan^{-1}\left(\sec\left(\cos^{-1}\left(\sin\frac{\alpha}{2}\right)\right)-1\right)$ is equal to :

(a) $\frac{\alpha}{4}$ (b) $\frac{\alpha}{2}$ (c) $\frac{\beta}{2}$ (d) $\frac{\beta}{4}$

7. The range of function $f(x)=\cot^{-1}(x^2+bx)$ is equal to :

(a) $\left(0,\frac{\pi}{4}\right]$ (b) $(0,\pi)$ (c) $\left(0,\frac{3\pi}{4}\right]$ (d) $\left[\frac{3\pi}{4},\pi\right)$

8. The number of solution(s) of the equation $|b|\sin^{-1}x=(a-b)x$, is equal to :

(a) 0 (b) 1 (c) 2 (d) 3

Paragraph for Question Nos. 9 and 10

Consider, $f(x)=\tan^{-1}\left(\frac{\sqrt{1-4x^2}-2\sqrt{3}x}{\sqrt{3-12x^2}+2x}\right)$

9. If $x\in\left(\frac{-\sqrt{3}}{4},\frac{1}{2}\right)$ then range of $f(x)$ is :

(a) $\left[0,\frac{\pi}{6}\right]$ (b) $\left[\frac{-\pi}{3},\frac{\pi}{6}\right)$ (c) $\left[\frac{-\pi}{3},0\right)$ (d) $\left[\frac{-\pi}{3},\frac{\pi}{2}\right)$

10. The value of $f\left(\frac{\sqrt{3}+1}{4\sqrt{2}}\right)+f\left(\frac{1}{2\sqrt{2}}\right)$ is equal to :

(a) $\frac{-5\pi}{12}$ (b) $\frac{-\pi}{12}$ (c) $\frac{-\pi}{3}$ (d) $\frac{5\pi}{12}$

Paragraph for Question Nos. 11 to 13

If α is the minimum value of the expression

$$y=\frac{4x^2-4x+17}{7+4x-4x^2} \text{ and } \beta=\left[\frac{1002\cdot1003\cdot1004\cdot1006\cdot1007\cdot1008}{\frac{1}{6}(1005)^6}\right]$$

where $[y]$ denotes largest integer less than or equal to y.

11. $\alpha+\beta$ is equal to :

(a) 8 (b) 7 (c) 5 (d) 2

12. If $k \in [\alpha, \beta]$ then least integral value of k for which $f(x)=\tan^{-1}(\sin(k\cos x))$ has its range $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$ is :

(a) -1 (b) 0 (c) 2 (d) 4

13. If $g(x)=\log_{\frac{11}{9}}(x+e^{\ln(\tan(\cot^{-1}(5+4x-x^2)))})$ and $x \in [\alpha, \beta]$ then range of $g(x)$ is :

(a) $(-\infty, \infty)$ (b) $[1, \infty)$ (c) $[-1, \infty)$ (d) $\left[\log_{\frac{11}{9}}\left(2+\frac{\pi}{2}\right), \infty\right)$

Paragraph for Question Nos. 14 to 16

Let $f(x)=\sin^{-1}\left(\frac{3x^4+6x^2-1}{(x^2+1)^3}\right)$ and $g(x)$ is defined as

$$g(x)=\begin{cases} f(x)+3\sec^{-1}(x^2+1) & |x|\le 1 \\ f(x)+3\,\mathrm{cosec}^{-1}(x^2+1) & |x|>1 \end{cases}$$

14. The value of $g\left(\sqrt{\tan 22\frac{1^\circ}{2}}\right)+g(\sqrt[3]{\cot 15^\circ})+g(\sqrt[3]{\sin 18^\circ})$ is equal to :

(a) 2π (b) 4π (c) $\frac{7\pi}{2}$ (d) $\frac{9\pi}{2}$

15. If $h(x)=\frac{1}{5}(3\sin x+4\cos x)$ then domain and range of $g(h(x))$ are respectively :

(a) $[-1, 1]; [\pi, 3\pi]$ (b) $R; \{\pi\}$ (c) $R; \left\{\frac{3\pi}{2}\right\}$ (d) $[-1, 1]; \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

16. Number of solutions of the equation $g(x)=\tan\left(\cot^{-1}\left(\left|\frac{1}{4x}\right|\right)\right)$ is :

(a) 0 (b) 1 (c) 2 (d) 4

Paragraph for Question Nos. 17 to 19

Let $f(x)=\frac{\pi}{4}+\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)-\tan^{-1}x$ and a_i ($a_i < a_{i+1}\ \forall\ i=1, 2, 3, \ldots, n$) be the positive integral values of x for which sgn $(f(x))=1$ where sgn$(\cdot)$ denotes signum function.

17. If $P(x)=x^2-4kx+3k^2$ is negative for all values of x lying in the interval (a_1, a_2) then set of real values of k is :

(a) $\left(\frac{1}{3}, 1\right)$ (b) $\left[\frac{2}{3}, 2\right)$ (c) $\left(\frac{1}{3}, 2\right)$ (d) $\left[\frac{2}{3}, 2\right]$

18. The function $g(x) = \dfrac{\tan\left(\left[\dfrac{3x}{x-(a_1+2)}\right]\pi\right)}{\sec^{-1}\left(\dfrac{2}{x-(a_1+a_2)}\right)+\cos^{-1}\left(\dfrac{x-(a_2+1)}{2}\right)}$, where $[\cdot]$ denotes greatest integer function, is discontinuous at $x =$ ....

(a) $a_1 - 1$ (b) $a_2 + 2$ (c) $a_2 - a_1$ (d) $4a_2 - a_1$

19. The number of points where $h(x) = \dfrac{x|(x-(a_1-1))(x-(a_2-2))|}{\sqrt{x^2-2a_2x+5}}$ is non-differentiable is/are :

(a) 0 (b) 1 (c) 2 (d) 3

EXERCISE - 3

More Than One Correct Answers

1. Let $f(x) = \sqrt{(\sin^{-1}x - \cos^{-1}x)}$, $g(x) = \sqrt{(\tan^{-1}x - \cot^{-1}x)}$ and $h(x) = \sqrt{(\sec^{-1}x - \text{cosec}^{-1}x)}$ then correct statement(s) is(are) :
   (a) Domain of $f(x) + g(x)$ is $\{1\}$
   (b) Domain of $g(x) + h(x)$ is $[\sqrt{2}, \infty)$
   (c) Domain of $h(x) + f(x)$ if $\left[\dfrac{1}{\sqrt{2}}, 1\right]$
   (d) Domain of $f(x) + g(x) + h(x)$ is ϕ

2. If $[y]$, 2 and x are the first three terms of a G.P. where x is an integer in the domain of $f(x) = \sec^{-1}x$ and $[k]$ denotes greatest integer less than or equal to k, then identify the correct statement(s) :
   (a) Number of such G.P.s are 3
   (b) Number of such G.P.s are 6
   (c) The set of values of y is $[-4, 5) - \{0\}$
   (d) The set of values of y is $[-4, -3) \cup [-2, 0) \cup [1, 3) \cup [4, 5)$

3. If $f : R \to \left[-\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ is a function defined by $f(x) = \tan^{-1}\left(x^4 - x^2 - \dfrac{7}{4} + \tan^{-1}\alpha\right)$ and f is surjective then :
   (a) $\cos^{-1}\left(\dfrac{1-\alpha^2}{1+\alpha^2}\right) = 2$ (b) $\alpha + \dfrac{1}{\alpha} = 2\,\text{cosec}\,2$
   (c) $\sin^{-1}\left(\dfrac{2\alpha}{\alpha^2+1}\right) = \pi - 2$ (d) $\tan^{-1}\left(\dfrac{2\alpha}{\alpha^2-1}\right) = 2 - \pi$

4. If $f(n) = \cot^{-1}(n+3) - 2\cot^{-1}(n+1) + \cot^{-1}(n-1), n \in N$, then $\sum_{n=1}^{\infty} f(n)$ is equal to :

(a) $\sum_{k=1}^{3} \tan^{-1}\left(\frac{1}{k}\right)$

(b) $\lim_{x \to \pi} \frac{-\sin(x)}{\pi - x}$

(c) $\lim_{x \to 0^-} \cot^{-1}\left(\frac{1}{x}\right)$

(d) $\lim_{x \to \pi} \frac{\pi(x-\pi)^2}{4(\cos x + 1)}$

5. Consider, $f(x) = \tan^{-1}\left(\frac{2}{x}\right)$, $g(x) = \sin^{-1}\left(\frac{2}{\sqrt{4+x^2}}\right)$ and $h(x) = \tan(\cos^{-1}(\sin x))$.

Identify the correct statement(s) :

(a) For $x > 0$, $(h(f(x)) + h(g(x)))$ is equal to $\frac{4}{x}$

(b) For $x < 0$, $(h(f(x)) + h(g(x)))$ is equal to 0

(c) For $x > 0$, $(h(f(x)) + h(g(x)))$ is equal to x

(d) For $x < 0$, $(h(f(x)) + h(g(x)))$ is equal to $\frac{x}{4}$

6. $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P. and x, y, z are also in A.P. $(y \neq 0, 1, -1)$ then :

(a) x, y, z are in G.P.

(b) x, y, z are in H.P.

(c) $x = y = z$

(d) $(x-y)^2 + (y-z)^2 + (z-x)^2 = 0$

7. Let $f(x) = \cot^{-1}(\text{sgn}(x)) + \sin^{-1}(x - \{x\})$ which of the following is(are) correct?

(a) Domain of $f(x)$ is $[-1, 2)$

(b) $f(x)$ is an even function $\forall x \in [-1, 1]$

(c) $f(x)$ is bounded

(d) Number of solution of the equation $f(x) = \frac{\pi}{2}$ is zero

[**Note :** $\{y\}$ and sgn (y) denotes fractional part of y and signum of y respectively.]

8. Which of the following functions represent identical graphs in x-y plane $\forall x \in [-1, 1]$?

(a) $f_1(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$

(b) $f_2(x) = \frac{\pi}{4} - \tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{1+x^2}}\right)$

(c) $f_3(x) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x^2$

(d) $f_4(x) = \frac{1}{2}\sin^{-1}x^2$

9. If $x^2 + 2x + n > 10 + \sin^{-1}(\sin 9) + \tan^{-1}(\tan 9)$ for all real x, then the possible value of n can be :

(a) 11 (b) 12 (c) 13 (d) 14

10. Let $\alpha = 3\cos^{-1}\left(\frac{5}{\sqrt{28}}\right) + 3\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\beta = 4\sin^{-1}\left(\frac{7\sqrt{2}}{10}\right) - 4\tan^{-1}\left(\frac{3}{4}\right)$, then which of the following does not hold(s) good?

(a) $\alpha < \pi$ but $\beta > \pi$
(b) $\alpha > \pi$ but $\beta < \pi$
(c) Both α and β are equal
(d) $\cos(\alpha + \beta) = 0$

11. If α is a real root of the equation $x^3 + 3x - \tan 2 = 0$ then $\cot^{-1}\alpha + \cot^{-1}\frac{1}{\alpha} - \frac{\pi}{2}$ can not be equal to :

(a) 0
(b) $\frac{\pi}{2}$
(c) π
(d) $\frac{3\pi}{2}$

12. If $\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sin^{-1}\left(\sqrt{1-\frac{x}{4}}\right) + \tan^{-1}y = \frac{2\pi}{3}$, then :

(a) maximum value of $x^2 + y^2$ is $\frac{49}{3}$
(b) maximum value of $x^2 + y^2$ is 4
(c) minimum value of $x^2 + y^2$ is $\frac{1}{3}$
(d) minimum value of $x^2 + y^2$ is 3

13. If $f(x) = \tan^{-1}\left(\frac{\sqrt{3}x - 3x}{3\sqrt{3} + x^2}\right) + \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$, $0 \le x \le 3$, then :

(a) the least value $f(x)$ is $-\frac{\pi}{3}$
(b) the greatest value of $f(x)$ is $\frac{\pi}{4}$
(c) the least value of $f(x)$ is 0
(d) the greatest value of $f(x)$ is $\frac{\pi}{3}$

14. If $2^{(\cos^{-1}x)^2}\sqrt{(\sin^{-1}y)^4 - 2(\sin^{-1}y)^2 + 2} = 1$, then the value of $(x - y)$ can be :

(a) $1 - \frac{\pi}{2}$
(b) $1 + \frac{\pi}{2}$
(c) $1 - \sin 1$
(d) $1 + \sin 1$

15. Consider a function $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ $-a\tan^{-1}x$ where a is any real constant. Find the value of 'a' if $f(x) = 0$ for all x :

(a) 6
(b) -6
(c) 2
(d) -2

16. Let $f(x) = \tan^{-1}(1 + x + x^2 + \ldots\ldots + \infty)$, $g(x) = \cot^{-1}\left(\frac{1}{1 - x + x^2 - x^3 + \ldots + \infty}\right)$, then find the correct statement $\forall\, x \in (0, 1)$:

(a) Range of $f(x) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(b) Range of $f(x) \in \left(0, \frac{\pi}{4}\right)$
(c) Range of $g(x) \in \left(\cot^{-1}2, \frac{\pi}{4}\right)$
(d) Range of $g(x) \in \left(\frac{\pi}{4}, \pi\right)$

17. Which of the following is(are) correct?

(a) Domain of $f(x) = \sin^{-1}(\cos^{-1}x + \tan^{-1}x + \cot^{-1}x)$ is null set
(b) Domain of $f(x) = \cos^{-1}(\tan^{-1}x + \cot^{-1}x + \sin^{-1}x)$ is $[-1, -\cos 1]$
(c) Domain of $f(x) = \sin^{-1}(\cos^{-1}x)$ is $[\cos 1, 1]$
(d) Domain of $f(x) = \cos^{-1}(\sin^{-1}x)$ is $[-\sin 1, \sin 1]$

EXERCISE - 4

Match the Columns Type

1. Column I **Column II**

(a) $2\cot(\cot^{-1}(3)+\cot^{-1}(7)+\cot^{-1}(13)+\cot^{-1}(21))$ has the value equal to — (p) 3

(b) If $\tan\left(\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+.....+\tan^{-1}\left(\frac{1}{381}\right)\right)$ $=\frac{m}{n}$, where $m, n \in N$, then the least value of $(m+n)$ is divisible by — (q) 4

(c) Number of integral ordered pairs (x, y) satisfying the equation $\text{arc}\tan\frac{1}{x}+\text{arc}\tan\frac{1}{y}=\text{arc}\tan\frac{1}{10}$, is — (r) 5

(d) The smallest positive integral value of n for which $(n-2)x^2+8x+n+4>\sin^{-1}(\sin 12)+\cos^{-1}(\cos 12)\ \forall\ x \in R$, is — (s) 8

(t) 10

2. Column I **Column II**

(a) Let $f: R \to R$ and $f_n(x)=f(f_{n-1}(x))\ \forall\ n \geq 2, n \in N$, the roots of equation $f_3(x)\,f_2(x)\,f(x)-25\,f_2(x)\,f(x)+175\,f(x)=375$. Which also satisfy equation $f(x)=x$ will be — (p) 1

(b) Let $f: [5, 10]$ onto $[4, 17]$, the integers in the range of $y=f(f(f(x)))$ is/are — (q) 5

(c) Let $f(x)=8\cot^{-1}(\cot x)+5\sin^{-1}(\sin x)+4\tan^{-1}(\tan x)-\sin(\sin^{-1}x)$, then possible integral values which $f(x)$ can take — (r) 10

(d) Let 'a' denotes the roots of equation $\cos(\cos^{-1}x)+\sin^{-1}\sin\left(\frac{1+x^2}{2}\right)=2\sec^{-1}(\sec x)$ then possible values of $[\,|10a|\,]$ where $[\cdot]$ denotes the greatest integer function will be — (s) 15

3. Let $f: R \to [\alpha, \infty)$, $f(x)=x^2+3ax+b$, $g(x)=\sin^{-1}\frac{x}{4}$ $(\alpha \in R)$.

Column I **Column II**

(a) The possible integral values of 'a' for which $f(x)$ is many one in interval $[-3, 5]$ is/are — (p) -2

(b) Let $a = -1$ and gof (x) is defined for $x \in [-1, 1]$ then possible integral values of b can be — (q) -1

(c) Let $a = 2$, $\alpha = -8$ the value(s) of b for which $f(x)$ is surjective is/are — (r) 0

(d) If $a = 1$, $b = 2$, then integers in the range of fog (x) is/are — (s) 1

4. If $S_n = \sum_{r=1}^{n} r!$ then for $n > 6$ $\left(\text{given } \sum_{r=1}^{6} r! = 873\right)$

Column I	Column II
(a) $\sin^{-1}\left(\sin\left(S_n - 7\left[\frac{S_n}{7}\right]\right)\right)$	(p) $5 - 2\pi$
(b) $\cos^{-1}\left(\cos\left(S_n - 7\left[\frac{S_n}{7}\right]\right)\right)$	(q) $2\pi - 5$
(c) $\tan^{-1}\left(\tan\left(S_n - 7\left[\frac{S_n}{7}\right]\right)\right)$	(r) $6 - 2\pi$
(d) $\cot^{-1}\left(\cot\left(S_n - 7\left[\frac{S_n}{7}\right]\right)\right)$	(s) $5 - \pi$
	(t) $\pi - 4$

(where [ ] denotes greatest integer function)

5.

Column I	Column II
(a) Let $f(x) = \sqrt{\log(\cos[\{x\}])}$, then $f(x)$ is	(p) Even and Periodic function
(b) Let $f : (-1, 1) \to R$ be defined as $f(x) = \sum_{r=1}^{100} [x^{2r}]$ then $f(x)$ is	(q) Bounded
(c) Let $f(x) = \cos^{-1}([e^x] - 1) + \sin^{-1}([e^x])$, then $f(x)$ is	(r) Domain contains at least one integer and atmost 3 integers
	(s) Both many one and odd function

[Note : $[y]$ and $\{y\}$ denote greatest integer and fractional part function of y respectively.]

6. Consider three functions, $f(x) = x^3 + x^2 + x + 1$, $g(x) = \frac{2x}{1-x^2}$ and $h(x) = \sin^{-1} x - \cos^{-1} x + \tan^{-1} x - \cot^{-1} x$.

Column I	Column II
(a) If range of $f(g(x))$ is $[a, b]$ then $(a + b)$ is equal to	(p) 1
(b) The number of integers in the range of $g(f(x))$ is	(q) 2
(c) The maximum value of $g(h(x))$ is equal to	(r) 3
(d) If the minimum value of $h(g(f(x)))$ is $\frac{k\pi}{2}$ then $\lvert k \rvert$ is equal to	(s) 4
	(t) 5

7. Consider $f(x) = \tan^{-1}\left(\frac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$ and m and M are respectively minimum and maximum values of $f(x)$ and $x = a$ $(a > 0)$ is the point in the domain of $f(x)$ where $f(x)$ attains its maximum value.

Column I	Column II
(a) If $\sin^{-1} 2\sqrt{x} = 3\tan^{-1}(\tan(m+M))$ then $8x$ equals	(p) 0
(b) If $\cos^{-1} x + \cos^{-1} y = 3\left\{\tan^{-1}\left(\tan\frac{7M}{2}\right) + \tan^{-1}\left(m + \tan\frac{3\pi}{8}\right)\right\}$ then $(x+y)$ equals	(q) 2
(c) The value of $\tan\left(\sec^{-1}\left(\frac{2}{a^2}\right) + M\right)$ equals	(r) -2
(d) If α and β are roots of the equation $x^2 - (\tan(3\sin^{-1}(\sin M)))x + a^4 = 0$, then $\alpha\beta - (\alpha+\beta)$ equals	(s) 1
	(t) -1

8. Let 'f' be a quadratic polynomial such that $f(-1-x) = f(-1+x)\ \forall\ x \in R$ and $(f(1)-5)^2 + (f(-1)-1)^2 = f'(-1)$.

Column I	Column II
(a) The value of $[\sin^{-1}(f(x))]$ whenever defined, is equal to	(p) 0
(b) The value of $[1 + \text{sgn}(f(x))]$ is equal to	(q) 1
(c) The value of $\left[\tan^{-1}\left(\frac{1}{f(x)}\right)\right]$ is equal to	(r) 2
(d) The value of $\left[2\cot^{-1}\left(\frac{1}{2^{f(x)}}\right)\right]$ is equal to	(s) 3

[Note : $[y]$ denotes greatest integer less than or equal to y.]

9. Match the entries of Column-I with one or more than one entries of column-II. Note that $[x]$, $\{x\}$ and sgn x denote largest integer less than or equal to x, fractional part of x and signum function of x respectively.

Column I	Column II
(a) Let $f : [-1, 1] \to R$ be defined by $f(x) = \sqrt[5]{x} + \sin^{-1} x$ then $f(x)$ is	(p) Odd
(b) Let $f : R \to \{-1, 0, 1\}$ be defined by $f(x) = \text{sgn}\left(\frac{1-\lvert x\rvert}{1+\lvert x\rvert}\right)$ then $f(x)$ is	(q) Even
(c) Let $f : [-4, 2] \to [0, 3]$ be defined by $f(x) = \sqrt{8-2x-x^2}$ then $f(x)$ is	(r) Onto

(d) Let $f:(-\infty, 0] \to [0, \infty)$ be defined by $f(x) = \dfrac{2^{-[x]}}{2^{[x]}} - 2^{|x|}$ then $f(x)$ is

(s) One-One

(t) Many-One

EXERCISE - 5

Integer Answer Type

1. If $x \in \left(0, \dfrac{\pi}{2}\right)$ satisfies the inequality $|\tan x - \sqrt{3}| + |4\sin^2 x - 3| + \left|\tan(\tan^{-1} x) - \dfrac{\pi}{3}\right| \le 0$, then find the value of $\left[\tan\left(\cot^{-1}\left(\dfrac{\sqrt{2}}{30x}\cos\left(\dfrac{3x}{4}\right)\right)\right)\right]$.

   [**Note :** [·] denotes greatest integer function.]

2. If x and y are positive integer satisfying $\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}\left(\dfrac{1}{y}\right) = \tan^{-1}\left(\dfrac{1}{7}\right)$ then find the number of ordered pairs of (x, y).

3. Consider $f(x) = \cos^{-1} x + \cos^{-1}\left(\dfrac{\sqrt{3}x + \sqrt{1-x^2}}{2}\right)$. If $\displaystyle\sum_{r=1}^{100} f\left[1 - \left(\frac{1}{10}\right)^r\right] = \frac{p}{q}\pi$ where p and q are relatively prime number, then find the value of $(p - 16q)$.

4. Find the number of ordered pairs (a, b) where $a, b \in R$ and satisfying
$$\frac{a^2 - 3a + \left[3 + \dfrac{1}{2+a^2}\right]}{\operatorname{sgn}(\cot^{-1} a)} = \frac{b^2 - 2b - \left[1 + \dfrac{1}{2+b^2}\right]b + 3}{\operatorname{sgn}(1+|b|)} = 1.$$

   [**Note :** $[k]$ denotes greatest integer less than or equal to k and sgn(k) denotes signum function of k.]

5. If the solution set of the inequality $\tan^{-1} x + \sin^{-1} x \ge \dfrac{\pi}{2}$ is $\left[\sqrt{\dfrac{\sqrt{\lambda} - 1}{\mu}}, 1\right]$, then find the value of $(\lambda + \mu)$.

6. Find the number of solutions of equation $\sin^{-1}(4\sin^2\theta + \sin\theta) + \cos^{-1}(-1 + 6\sin\theta) = \dfrac{\pi}{2}$, in $\theta \in [0, 5\pi]$.

7. The value of $\cot\left(\displaystyle\sum_{k=1}^{10} \cot^{-1}(1 + k + k^2)\right) = \dfrac{a}{b}$ where a and b are coprime, find the value of $(a + b)$.

8. If $\displaystyle\sum_{r=1}^{10} \tan^{-1}\left(\frac{3}{9r^2 + 3r - 1}\right) = \cot^{-1}\left(\frac{m}{n}\right)$ (where m and n are coprime), then find $(2m + n)$.

9. Let S be set of domain of the function $f(x) = \sqrt{\frac{\pi}{2} - \tan^{-1}\sqrt{-x^2 + 5x - 6}}$. If $\lambda = \alpha + \frac{1}{\alpha}$ where $\alpha \in S$ and λ is an integer then find the value of (λ^2).

10. If the $\lim_{n\to\infty} \sum_{k=2}^{n} \cos^{-1}\left(\frac{1+\sqrt{(k-1)k(k+1)(k+2)}}{k(k+1)}\right) = \frac{120\pi}{k}$ then find the value of k.

11. Consider, $f(x) = |x-1| + |2x-\pi| + |x-3|$ and $g(x) = \sin^{-1}x + \tan^{-1}x$. The value of x for which $f(g(x))$ is minimum is $\sqrt{k\sin\frac{\pi}{\lambda}}$ where k and $\lambda \in N$ then find the value of $(k+\lambda)$.

12. Let $f(x) = \sin^{-1}x + \cos^{-1}x^2 + \sin^{-1}x^3 + \cos^{-1}x^4 + \ldots\ldots + \sin^{-1}x^{2n-1} + \cos^{-1}x^{2n}$ and $g(x) = \cos^{-1}x + \sin^{-1}x^2 + \cos^{-1}x^3 + \sin^{-1}x^4 + \ldots\ldots + \cos^{-1}x^{2n-1} + \sin^{-1}x^{2n}$, where $n \in N$. If sum of least value of $f(x)$ and greatest value of $g(x)$ be 8π then the value of n will be.

13. If $x^2 + ax + b = 0$ has two distinct negative integral roots and $\left(\log_{1/2}\left(\frac{2}{\pi}\cot^{-1}x + 1\right)\right)^2 + a\log_{1/2}\left(1 - \frac{\tan^{-1}x}{\pi}\right) = a - b$ has no real solution. Then find the minimum value of a.

14. If $\alpha = \sum_{n=1}^{\infty} \tan^{-1}\left\{\frac{16(2n+1)}{(2n+1)^4 - 4(2n+1)^2 + 16}\right\}$ and $\sin\alpha = \frac{a}{b}$ where $a, b \in N$, then find the least value of $a^2 + b^2$.

15. If $\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$, (where $x, y > 0$) then find the maximum value of $9x^2 - 12xy\cos\theta + 4y^2$.

16. For $x, y, z, t \in R$, if $\sin^{-1}x + \cos^{-1}y + \sec^{-1}z \geq t^2 - \sqrt{2\pi}t + 3\pi$, then find the value of $\sec\left(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z + \tan^{-1}\left(\sqrt{\frac{2}{\pi}}t\right)\right)$.

17. Let α and β be the roots of the equation $3x^2 - [2a + 4\sin^{-1}(\sqrt{[a]+[-a]})]x + \sqrt{3-|a|} = 0$, $a \in R^+$ (the set of all positive real numbers). Find the sum of all values of a for which $\alpha, \beta \in R$ (the set of all real numbers).

[Note : $[k]$ denotes the largest integer less than or equal to k.]

18. If range of the function $f(x) = \cot^{-1}\left(\frac{x^2}{x^2+1}\right)$ is $(a, b]$ then find the value of $\frac{b}{a}$.

19. Let $f(x) = \begin{cases} \cos^{-1}\left(\dfrac{1+x}{\sqrt{2(1+x^2)}}\right), & x \le 0 \\ \tan^{-1} x, & x > 0 \end{cases}$. If the range of values of k for which the equation $f(x) = k$ has exactly two solutions is $[a, b)$ then find the value of $\left(\dfrac{1}{a} + \dfrac{1}{b}\right)\pi$.

20. If $f(x) = \sin^{-1}\left[\dfrac{x^2(2x^2+1)-1}{1+x^2}\right] = \begin{cases} a\pi + b\cos^{-1} x, & 0 \le x \le 1 \\ p\pi + q\sin^{-1} x, & -1 \le x < 0 \end{cases}$, then find the absolute of $(a + b + p + q)$.

21. Let $f(x) = \dfrac{1}{\pi}(\sin^{-1} x + \cos^{-1} x + \tan^{-1} x) + \dfrac{x+1}{x^2+2x+10}$. If the absolute maximum value of $f(x)$ is M. Find $52M$.

22. If the sum $\displaystyle\sum_{n=1}^{10}\sum_{m=1}^{10} \tan^{-1}\left(\frac{m}{n}\right) = k\pi$, find the value of k.

23. Let m be the number of solutions of $\sin 2x + \cos 2x + \cos x + 1 = 0$ in $x \in \left(0, \dfrac{\pi}{2}\right)$ and $n = \sin\left[\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right) + \cos^{-1}\left(\cos\dfrac{7\pi}{3}\right)\right]$, then find the value of $(m + n)$.

24. Let $f(n) = \displaystyle\sum_{k=-n}^{n}\left(\cot^{-1}\frac{1}{k} - \tan^{-1} k\right)$, $k \ne 0$. If the value of $\displaystyle\sum_{n=2}^{10}[f(n) + f(n-1)]$ is $m\pi$ then find the value of m.

25. If $f(x) = x^3 - 3x + \sin^{-1}(a^2 - 3a + 2)$. Then the smallest positive integer 'a' for which $f(x) = 0$ has three distinct real solution.

26. Let $S_n = \displaystyle\sum_{n=1}^{n}\sin^{-1}\left[\frac{(2n+1)}{n(n+1)(\sqrt{n(n+2)} + \sqrt{(n+1)(n-1)})}\right]$. Find the value of $100\cos(S_{99})$.

27. Let $f(x) = x^2 - 2ax + a - 2$ and $g(x) = \left[2 + \sin^{-1}\dfrac{2x}{1+x^2}\right]$. If the set of real values of 'a' for which $f[g(x)] < 0 \ \forall\ x \in R$ is (k_1, k_2) then find the value of $(10k_1 + 3k_2)$.
[**Note :** $[k]$ denotes greatest integer less than or equal to k.]

28. Consider $f(x) = \sin^{-1}\left(\dfrac{x+3}{2x+5}\right)$, $g(x) = \sin^{-1}\left(\dfrac{ax^2+b}{x^2+5}\right)$. If $\displaystyle\lim_{x\to\infty}(f(x) - g(x)) = 0$ and $\displaystyle\lim_{x\to 0}(f(x) + g(x)) = \frac{\pi}{4}$, then find the value of $(a + b^2)$.

29. If range of the function $f(x) = \left(\cos^{-1}\dfrac{x}{2}\right)^2 + \pi\sin^{-1}\dfrac{x}{2} - \left(\sin^{-1}\dfrac{x}{2}\right)^2 + \dfrac{\pi^2}{12}(x^2 + 6x + 8)$ is $[a\pi^2, b\pi^2]$, then find the value of $2(a + b)$.

30. Consider, $\alpha = \sin^{-1}\left(\frac{2x}{1+x^2}\right), x \in [-1, 1], \beta = \cos^{-1}\left(\frac{3\cos y - 4\sin y}{10}\right), y \in [0, 2\pi]$ and $\gamma = 2\tan^{-1}(z^2 - 4z + 5), z \in R$. If α, β and γ are interior angles of a triangle such that $(\beta + \gamma)$ is minimum then $x + \tan y + z = \frac{a - \sqrt{b}}{c}$ where, $a, b, c \in N$. Find the least value of $(a + b + c)$.

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (a)	2. (b)	3. (a)	4. (d)	5. (c)	6. (d)	7. (b)	8. (b)	9. (d)	10. (c)
11. (d)	12. (a)	13. (b)	14. (b)	15. (b)	16. (b)	17. (d)	18. (a)	19. (c)	20. (c)
21. (d)	22. (a)	23. (d)	24. (a)	25. (d)	26. (b)	27. (c)	28. (a)	29. (d)	30. (a)
31. (b)	32. (a)	33. (b)	34. (d)	35. (b)	36. (c)	37. (c)	38. (d)	39. (a)	40. (d)
41. (a)	42. (a)	43. (a)	44. (b)	45. (a)	46. (c)	47. (c)	48. (c)		

EXERCISE 2 : Linked Comprehension Type

1. (b)	2. (b)	3. (c)	4. (c)	5. (c)	6. (a)	7. (c)	8. (d)	9. (d)	10. (c)
11. (b)	12. (c)	13. (b)	14. (c)	15. (b)	16. (d)	17. (d)	18. (d)	19. (a)	

EXERCISE 3 : More Than One Correct Answers

1. (a, b, d)	2. (b, d)	3. (a, b, c)	4. (a, d)	5. (b, c)
6. (a, b, c, d)	7. (a, c)	8. (a, b, c, d)	9. (b, c, d)	10. (a, b, d)
11. (a, b, d)	12. (a, c)	13. (b, c)	14. (c, d)	15. (a, c, d)
16. (a, c)	17. (a, b, c, d)			

EXERCISE 4 : Match the Columns Type

1. (a) (p), (b) (q) (r) (s) (t), (c) (q), (d) (r)
2. (a) (q) (s), (b) (q) (r) (s), (c) (p) (q) (r) (s), (d) (p) (r)
3. (a) (p) (q) (r) (s), (b) (p) (q) (r), (c) (s), (d) (r) (s)
4. (a) (p), (b) (q), (c) (p), (d) (s)
5. (a) (p) (q) (s), (b) (q) (r) (s), (c) (q)
6. (a) (s), (b) (r), (c) (p), (d) (t)
7. (a) (s), (b) (r), (c) (s), (d) (q)
8. (a) (q), (b) (r), (c) (p), (d) (s)
9. (a) (p) (s), (b) (q) (r) (t), (c) (r) (t), (d) (t)

EXERCISE 5 : Integer Answer Type

1. 31	2. 6	3. 2	4. 4	5. 7
6. 6	7. 11	8. 32	9. 9	10. 720
11. 12	12. 8	13. 5	14. 41	15. 36
16. 1	17. 5	18. 2	19. 6	20. 4
21. 47	22. 25	23. 1	24. 99	25. 1
26. 1	27. 20	28. 1	29. 5	30. 38

Chapter 3

Limits, Continuity and Differentiability

KEY CONCEPTS

LIMIT

1. Limit of a function $f(x)$ is said to exist as, $x \to a$ when

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ = finite quantity.

2. INDETERMINANT FORMS :

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, 0^{\circ}, \infty^{\circ}, \infty - \infty$ and 1^{∞}

Note : (a) We cannot plot ∞ on the paper. Infinity (∞) is a symbol and not a number. It does not obey the laws of elementary algebra.

(b) $\infty + \infty = \infty$

(c) $\infty \times \infty = \infty$

(d) $(a/\infty) = 0$ if a is finite

(e) $\frac{a}{0}$ is not defined, if $a \neq 0$.

(f) $ab = 0$, if and only if $a = 0$ or $b = 0$ and a and b are finite.

3. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$. If l and m exists then :

(a) $\lim_{x \to a} f(x) \pm g(x) = l \pm m$

(b) $\lim_{x \to a} f(x) \cdot g(x) = l . m$

(Remember : $\lim_{x \to a} \Rightarrow x \neq a$)

(c) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$, provided $m \neq 0$

(d) $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$; where k is a constant.

(e) $\lim_{x\to a} f[g(x)] = f(\lim_{x\to a} g(x)) = f(m)$; provided f is continuous at $g(x) = m$.

For example $\lim_{x\to a} \ln(f(x)) = \ln\left[\lim_{x\to a} f(x)\right] \ln l \; (l > 0)$.

4. SQUEEZE PLAY THEOREM :

If $f(x) \le g(x) \le h(x)\ \forall\ x$ and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = l$.

5. STANDARD LIMITS :

(a) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\tan^{-1} x}{x} = \lim_{x\to 0} \frac{\sin^{-1} x}{x}$

[Where x is measured in radians]

(b) $\lim_{x\to 0} (1+x)^{1/x} = e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ note however there $\lim_{\substack{h\to 0\\ n\to\infty}} (1-h)^n = 0$ and

$\lim_{\substack{h\to 0\\ n\to\infty}} (1+h)^n \to \infty$

(c) If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} \phi(x) = \infty$, then; $\lim_{x\to a} [f(x)]^{\phi(x)} = e^{\lim_{x\to a} \phi(x)(f(x)-1)}$

(d) If $\lim_{x\to a} f(x) = A > 0$ and $\lim_{x\to a} \phi(x) = B$ (a finite quantity) then; $\lim_{x\to a} (f(x))^{\phi(x)} = e^z$

where $z = \lim_{x\to a} \phi(x).\ln(f(x)) = e^{B\ln A} = A^B$

(e) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a\ (a > 0)$. In particular $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(f) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

6. The following strategies should be born in mind for evaluating the limits:

(a) Factorisation

(b) Rationalisation or double rationalisation

(c) Use of trigonometric transformation; appropriate substitution and using standard limits

(d) Expansion of function like Binomial expansion, exponential and logarithmic expansion, expansion of $\sin x$, $\cos x$, $\tan x$ should be remembered by heart and are given below :

(i) $a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2\ln^2 a}{2!} + \frac{x^3\ln^3 a}{3!} + \ldots\ a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots;\ x \in R;$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $-1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

CONTINUITY

1. A function $f(x)$ is said to be continuous at $x = c$, if $\lim_{x \to c} f(x) = f(c)$. Symbolically f is continuous at $x = c$ if $\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$.

i.e. LHL at $x = c$ = RHL at $x = c$ equals value of 'f' at $x = c$. It should be noted that continuity of a function at $x = a$ is meaningful only if the function is defined in the immediate neighbourhood of $x = a$, not necessarily at $x = a$.

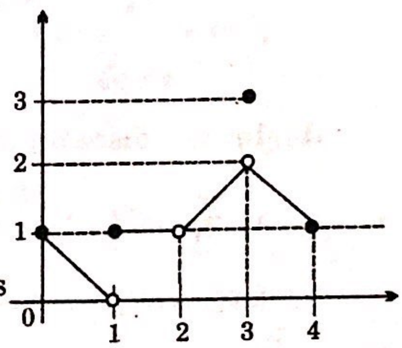
2. REASONS OF DISCONTINUITY:

(a) $\lim_{x \to c} f(x)$ does not exist, *i.e.*, $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

(b) $f(x)$ is not defined at $x = c$

(c) $\lim_{x \to c} f(x) \neq f(c)$

Geometrically, the graph of the function will exhibit a break at $x = c$. The graph as shown is discontinuous at $x = 1, 2$ and 3.



3. TYPES OF DISCONTINUITIES :

Type-1 : (Removable Type of Discontinuities)

In case $\lim_{x \to c} f(x)$ exists but is not equal to $f(c)$ then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim_{x \to c} f(x) = f(c)$ and make it continuous at $x = c$.

Removable type of discontinuity can be further classified as :

(a) Missing Point Discontinuity : Where $\lim_{x\to a} f(x)$ exists finitely but $f(a)$ is not defined, e.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at $x = 1$, and $f(x) = \frac{\sin x}{x}$ has a missing point discontinuity at $x = 0$.

(b) Isolated Point Discontinuity : Where $\lim_{x\to a} f(x)$ exists and $f(a)$ also exists but; $\lim_{x\to a} \neq f(a)$. e.g. $f(x) = \frac{x^2-16}{x-4}$, $x \neq 4$ and $f(4) = 9$ has an isolated point discontinuity at $x = 4$. Similarly $f(x) = [x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$ has an isolated point discontinuity at all $x \in I$.

Type-2 : (Non-removable Type of Discontinuities)

In case $\lim_{x\to c} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such discontinuities are known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

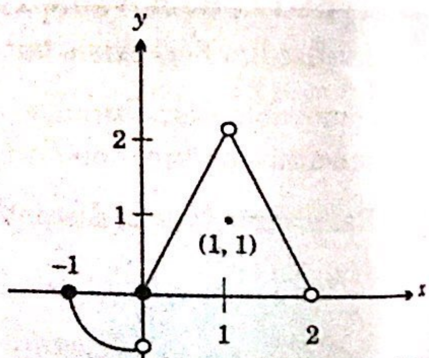
(a) Finite discontinuity e.g. $f(x) = x - [x]$ at all integral x; $f(x) = \tan^{-1}\frac{1}{x}$ at $x = 0$ and $f(x) = \frac{1}{1+2^{\frac{1}{x}}}$ at $x = 0$ (note that $f(0^+) = 0; f(0^-) = 1$)

(b) Infinite discontinuity e.g. $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at $x = 4$; $f(x) = 2^{\tan x}$ at $x = \frac{\pi}{2}$ and $f(x) = \frac{\cos x}{x}$ at $x = 0$.

(c) Oscillatory discontinuity e.g. $f(x) = \sin\frac{1}{x}$ at $x = 0$.

In all these cases the value of $f(a)$ of the function at $x = a$ (point of discontinuity) may or may not exist but $\lim_{x\to a}$ does not exist.

Note : From the adjacent graph note that
- f is continuous at $x = -1$
- f has isolated discontinuity at $x = 1$
- f has missing point discontinuity at $x = 2$
- f has non removable (finite type) discontinuity at the origin.



Nature of discontinuity

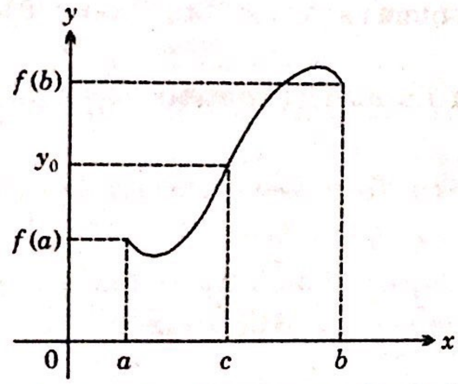
4. In case of discontinuity of the second kind the non-negative difference between the value of the RHL at $x = c$ and LHL at $x = c$ is called **The Jump of Discontinuity.** A function having a finite number of jumps in a given interval I is called a **Piece Wise Continuous** or **Sectionally Continuous** function in this interval.

5. All Polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their domains.

6. If f and g are two functions that are continuous at $x = c$ then the functions defined by : $F_1(x) = f(x) \pm g(x)$; $F_2(x) = Kf(x)$, K any real number; $F_3(x) = f(x) \cdot g(x)$ are also continuous at $x = c$. Further, if $g(c)$ is not zero, then $F_4(x) = \dfrac{f(x)}{g(x)}$ is also continuous at $x = c$.

7. THE INTERMEDIATE VALUE THEOREM

Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b), their exists a number c between a and b such that $f(c) = y_0$.



The function f, being continuous on $[a, b)$ takes on every value between $f(a)$ and $f(b)$

Note : (a) If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

$$f(x) = x \text{ and } g(x) = \begin{cases} \sin\dfrac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(b) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$. e.g.,

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(c) A continuous function whose domain is closed must have a range also in closed interval.

(d) If f is continuous at $x = c$ and g is continuous at $x = f(c)$ then the composite $g[f(x)]$ is continuous at $x = c$. e.g. $f(x) = \frac{x \sin x}{x^2 + 2}$ and $g(x) = |x|$ are continuous at $x = 0$, hence the composite $(gof)(x) = \left|\frac{x \sin x}{x^2 + 2}\right|$ will also be continuous at $x = 0$.

8. CONTINUITY IN AN INTERVAL :

(a) A function f is said to be continuous in (a, b) if f is continuous at each and every point $\in (a, b)$.

(b) A function f is said to be continuous in a closed interval $[a, b]$ if :

(i) f is continuous in the open interval (a, b)

(ii) f is right continuous at 'a' i.e. $\lim_{x \to a^+} f(x) = f(a) =$ a finite quantity.

(iii) f is left continuous at 'b' i.e. $\lim_{x \to b^-} f(x) = f(b) =$ a finite quantity.

Note : A function f which is continuous in $[a, b]$ possesses the following properties:

(a) If f(a) and f(b) possess opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).

(b) If K is any real number between f(a) and f(b), then there exists at least one solution of the equation $f(x) = K$ in the open interval (a, b).

9. SINGLE POINT CONTINUITY :

Functions which are continuous only at one point are said to exhibit single point continuity e.g. $f(x) = \begin{bmatrix} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{bmatrix}$ and $g(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{bmatrix}$ are both continuous only at $x = 0$.

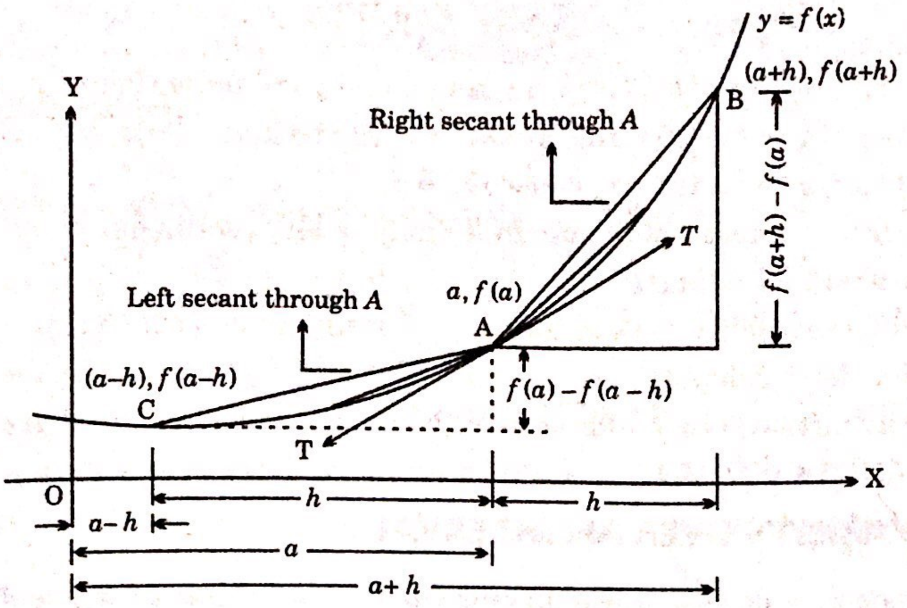
DIFFERENTIABILITY

1. RIGHT HAND AND LEFT HAND DERIVATIVES :

By definition : $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ if it exist

(a) The right hand derivative of f' at $x = a$ denoted by $f'(a^+)$ is defined by :

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h},$$ provided the limit exists and is finite.



(b) The left hand derivative : of f at $x = a$ denoted by $f'(a^+)$ is defined by :

$f'(a^-) = \lim_{h \to 0^+} \dfrac{f(a-h) - f(a)}{-h}$, provided the limit exists and is finite. We also write $f'(a^+) = f'_+(a)$ and $f'(a^-) = f'_-(a)$.

- This geometrically means that a unique tangent with finite slope can be drawn at $x = a$ as shown in the figure.

(c) Derivability and Continuity :

(i) If $f'(a)$ exists then $f(x)$ is derivable at $x = a \Rightarrow f(x)$ is continuous at $x = a$.

(ii) If a function f is derivable at x then f is continuous at x.

For : $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists.

Also $f(x+h) - f(x) = \dfrac{f(x+h) - f(x)}{h} \cdot h \, [h \neq 0]$

Therefore :

$$\lim_{h \to 0} [f(x+h) - f(x)] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot h = f'(x) \cdot 0 = 0$$

Therefore

$$\lim_{h \to 0} [f(x+h) - f(x)] = 0 \quad \Rightarrow \lim_{h \to 0} f(x+h) = f(x) \quad \Rightarrow f \text{ is continuous at } x.$$

Note : If $f(x)$ is derivable for every point of its domain of definition, then it is continuous in that domain.

The converse of the above result is not true :

"IF f IS CONTINUOUS AT x, THEN f IS DERIVABLE AT x" IS NOT TRUE.

e.g. the functions $f(x) = |x|$ and $g(x) = x \sin \dfrac{1}{x}$; $x \neq 0$ and $g(0) = 0$ are continuous at $x = 0$ but not derivable at $x = 0$.

Note :

(a) Let $f'_+(a) = p$ and $f'_-(a) = q$ where p and q are finite then :

(i) $p = q \Rightarrow f$ is derivable at $x = a \Rightarrow f$ is continuous at $x = a$.

(ii) $p \neq q \Rightarrow f$ is not derivable at $x = a$.

It is very important to note that f may be still continuous at $x = a$.

In short, for a function f:

Differentiability $\Rightarrow$ Continuity; Continuity $\not\Rightarrow$ derivability;

Non derivability $\not\Rightarrow$ discontinuous; But discontinuity $\Rightarrow$ Non derivability

(b) If a function f is not differentiable but is continuous at $x = a$ it geometrically implies a sharp corner at $x = a$.

2. DERIVABILITY OVER AN INTERVAL :

$f(x)$ is said to be derivable over an interval if it is derivable at each and every point of the interval $f(x)$ is said to be derivable over the closed interval $[a, b]$ if :

(a) for the points a and b, $f'(a+)$ and $f'(b-)$ exist and

(b) for any point c such that $a < c < b$, $f'(c+)$ and $f'(c-)$ exist and are equal.

Note :

(a) If $f(x)$ and $g(x)$ are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$ will also be derivable at $x = a$ and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$.

(b) If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$, *e.g.*, $f(x) = x$ and $g(x) = |x|$.

(c) If $f(x)$ and $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$, *e.g.*, $f(x) = |x|$ and $g(x) = |x|$.

(d) If $f(x)$ and $g(x)$ both are non-derivative at x = a then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function, *e.g.*, $f(x) = |x|$ and $g(x) = -|x|$.

(e) If $f(x)$ is derivable at $x = a \not\Rightarrow f'(x)$ is continuous at $x = a$.

$$e.g.\ f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(f) **A Surprising Result :** Suppose that the function $f(x)$ and $g(x)$ defined in the interval (x_1, x_2) containing the point x_0, and if f is differentiable at $x = x_0$ with $f(x_0) = 0$ together with g is continuous as $x = x_0$ then the function $F(x) = f(x) \cdot g(x)$ is differentiable at $x = x_0$, *e.g.* $F(x) = \sin x \cdot x^{2/3}$ is differentiable at $x = 0$.

EXERCISE - 1

Only One Correct Answer

1. If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$ then

$\lim_{x\to\alpha} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ equals :

(a) 0 (b) $\frac{1}{2}(\alpha-\beta)^2$ (c) $\frac{a^2}{2}(\alpha-\beta)^2$ (d) $-\frac{a^2}{2}(\alpha-\beta)^2$

2. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC and P be the perimeter of ABC then $\lim_{h\to 0} \frac{\Delta}{P^3}$ equals (where Δ is the area of the triangle) :

(a) $\frac{1}{32r}$ (b) $\frac{1}{64r}$ (c) $\frac{1}{128r}$ (d) none of these

3. Let $f : R \to R$ be a continuous function such that $f(x) = 19\ \forall\ x \in Q$ and $g(x) = \sqrt{39 + x^2}$, then the value of $g(f(x))$, is

(a) 18 (b) 20 (c) 25 (d) 17

4. Let f be an injective function with domain $[a, b]$ and range $[c, d]$. If α is a point in (a, b) such that f has left hand derivative l and right hand derivative r at $x = \alpha$ with both l and r non-zero different and negative, then left hand derivative and right hand derivative of f^{-1} at $x = f(\alpha)$ respectively, is :

(a) $\frac{1}{r}, \frac{1}{l}$ (b) r, l (c) $\frac{1}{l}, \frac{1}{r}$ (d) l, r

5. If $f(x) = \max.\left(x^4, x^2, \frac{1}{81}\right) \forall\ x \in [0, \infty)$, then the sum of the square of reciprocal of all the values of x where $f(x)$ is non-differentiable, is equal to :

(a) 1 (b) 81 (c) 82 (d) $\frac{82}{81}$

6. Let $y = f(x)$ be an even continuous function on R whose graph is passing through the point (1, 1). If $\lim_{x\to 0} f(x) = 2$ and $g(x)$ be a function defined on R such that $\lim_{x\to 0} g(x) = 3$, then which one of the following statement is incorrect?

(a) $\lim_{x\to 0} f(x-1) = 1$ (b) $\lim_{x\to 0} g(-x) = 3$

(c) $\lim_{x\to 0} [f(2x) + g(-x)] = 5$ (d) $\lim_{x\to 0} [5f(x-1) - 2g(-x)] = 1$

7. If $f(x)$ is a differentiable function such that $f'(2) = 6$ and $f'(1) = 4$, then $\lim_{h\to 0} \frac{f(h^3 + 3h + 2) - f(2)}{f(2h - 2h^2 + 1) - f(1)}$ equals :

(a) $\frac{4}{3}$ (b) $\frac{4}{9}$ (c) $\frac{3}{4}$ (d) $\frac{9}{4}$

8. Let $P(x) = a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_{100}x^{100}$, where $a_1 = 1$ and $a_i \in R\ \forall\ i = 2, 3, 4, \ldots,$ 100, then $\lim_{x\to 0} \frac{\sqrt[100]{1 + P(x)} - 1}{x}$ has the value equal to :

(a) 100 (b) $\frac{1}{100}$ (c) 1 (d) 5050

9. Let $f(x) = \frac{\sqrt{\log(x^2 + kx + k + 1)}}{x^2 + k}$. If $f(x)$ is continuous for all $x \in R$, then the range of k is :

(a) $(-\infty, 0) \cup [4, \infty)$ (b) $(0, 4]$

(c) $[0, 4)$ (d) $[0, 4]$

10. Let $f(x) = \max.\left\{\left|x^2 - 2|x|\right|, |x|\right\}$ and $g(x) = \min.\left\{\left|x^2 - 2|x|\right|, |x|\right\}$ then :

(a) both $f(x)$ and $g(x)$ are non differentiable at 5 points

(b) $f(x)$ is not differentiable at 5 points whether $g(x)$ is non differentiable at 7 points

(c) number of points of non differentiability for $f(x)$ and $g(x)$ are 7 and 5 respectively

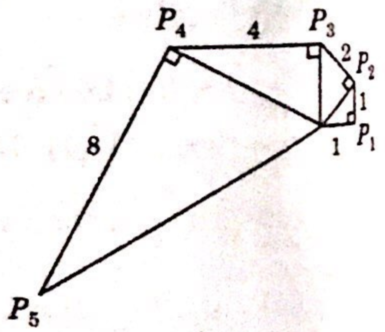
(d) both $f(x)$ and $g(x)$ are non differentiable at 3 and 5 points respectively

11. If $\lim_{x\to a}\left[\sin^{-1}\frac{2x}{1+x^2}\right]$ does not exist, then the number of possible values of a, is :

[Note: $[k]$ denotes the greatest integer less than or equal to k.]

(a) 2 (b) 3 (c) 4 (d) 5

12. Let $\{P_n\}$ be a sequence of points determined as in the figure. Thus $|AP_1| = 1$, $|P_nP_{n+1}| = 2^{n-1}$, and angle AP_nP_{n+1} is a right angle. $\lim_{n\to\infty} \angle P_nAP_{n+1}$ equals :



(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{12}$

13. Let α and β be the roots of the equation, $ax^2 + bx + c = 0$ where $1 < \alpha < \beta$,

then $\lim_{x\to m} \frac{|ax^2 + bx + c|}{ax^2 + bx + c} = 1$ when :

(a) $a > 0$ and $m > 1$

(b) $a < 0$ and $m < 1$

(c) $a < 0$ and $\alpha < m < \beta$

(d) $\frac{|a|}{a} = 1$ and $m > \alpha$

14. If f is a differentiable function on R such that $f(x+y), f(x)f(y), f(x-y)$ (taken in that order) are in arithmetic progression for all $x, y \in R$ and $f(0) \neq 0$, then :

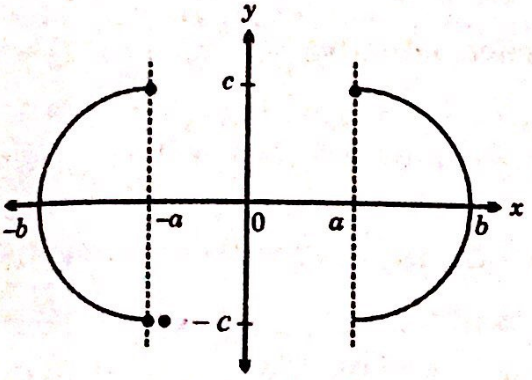
(a) $f'(0) = -1$

(b) $f'(0) = 1$

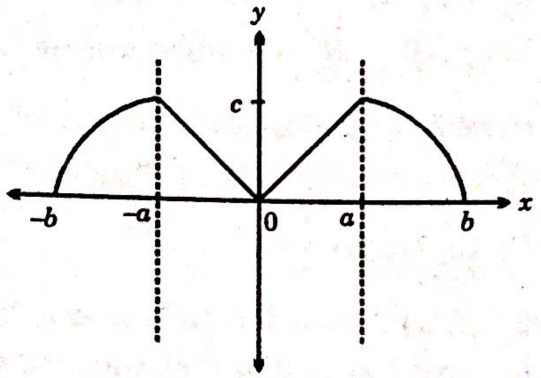
(c) $f'(1) - f'(-1) = 0$

(d) $f'(1) + f'(-1) = 0$

15. If graphs of $|y| = f(x)$ and $y = |f(x)|$ are given as below ($a, b > 0$).



$|y| = f(x)$



$y = |f(x)|$

Then identify the correct statement.

(a) $f(x)$ is discontinuous at 2 points in $[-b, b]$ and non-differentiable at 2 points in $(-b, b)$.

(b) $f(x)$ is discontinuous at 2 points in $[-b, b]$ and non-differentiable at 3 points in $(-b, b)$.

(c) $f(x)$ is discontinuous at 3 points in $[-b, b]$ and non-differentiable at 3 points in $(-b, b)$.

(d) $f(x)$ is discontinuous at 3 points in $[-b, b]$ and non-differentiable at 4 points in $(-b, b)$.

16. If $\alpha_1, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ are the roots of equation $x^{n+1} - 5x^2 + 6x - 3 = 0$ and $(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)\ldots.(\alpha_1 - \alpha_n) = k$, then $n(n+1)\alpha_1^{n-1} - 2k$ equals :

(a) 5 (b) 10 (c) 12 (d) 5

17. Let $f(x) = \cos 2x \cdot \cos 4x \cdot \cos 6x \cdot \cos 8x \cdot \cos 10x$, then $\lim_{x \to 0} \frac{1-(f(x))^3}{5\sin^2 x}$ equals :

(a) 660 (b) 135 (c) 132 (d) 66

18. If $\lim_{x \to 0}\left(\left[\frac{\sin^{-1} x}{x}\right] + \left[\frac{2^2 \sin^{-1} 2x}{x}\right] + \left[\frac{3^2 \sin^{-1} 3x}{x}\right] + \ldots\ldots + \left[\frac{n^2 \sin^{-1} nx}{x}\right]\right) = 100$, then the value of n, is :

[**Note :** $[k]$ denotes the greatest integer less than or equal to k.]

(a) 2 (b) 3 (c) 4 (d) 5

19. Let $f : R \to R$ be defined by $f(x) = \begin{cases} \{x\}, & \text{if} \quad x \in Q \\ x, & \text{if} \quad x \in R - Q \end{cases}$

If $\lim_{x \to \alpha} f(x)$ exist, then the true set of values of α is

[**Note :** $\{k\}$ denotes the fractional part of k and Q be the set of all rational numbers.]

(a) $(-1, 1)$ (b) $(-1, 0]$ (c) $(0, 1)$ (d) $[0, 1)$

20. Let $f : R \to R$ be a differentiable function satisfying $2f\left(\frac{x+y}{2}\right) - f(y) = f(x)$" $x, y \in R$.

If $f(0) = 5$ and $f'(5) = -1$, then :

(a) $\lim_{x \to 4} (f(x))^{\frac{1}{x-4}} = e$

(b) $f(|x|)$ is non-derivable at exactly 2 points

(c) area bounded by $f(x)$, x-axis and y-axis is 25 sq. units

(d) $|f(|x|)|$ is non-derivable at exactly 3 points

21. If $\lim_{n \to \infty} \frac{4^n}{(4 - \sqrt{3} + 2\sin\theta)^{n+2}}$ exists and is equal to p $(p \neq 0)$ where $\theta \in \left(0, \frac{\pi}{2}\right)$, then the value of $\left(\frac{p + \cos\theta}{p}\right)$ is equal to :

(a) 7 (b) 8 (c) 9 (d) 10

22. Given $f(x) = \begin{cases} \log_a (a|[x] + [-x]|)^x \left(\dfrac{a^{\left(\frac{[x]+[-x]}{|x|}\right)^{\frac{2}{-5}}}}{3 + a^{\frac{1}{|x|}}} \right) & \text{for} \quad |x| \neq 0; a > 1 \\ 0 & \text{for} \quad x = 0 \end{cases}$

where [ ] represents the integral part function, then :

(a) f is continuous but not differentiable at $x = 0$

(b) f is continuous and differentiable at $x = 0$

(c) the differentiability of 'f' at $x = 0$ depends on the value of a

(d) f is continuous and differentiable at $x = 0$ and for $a = e$ only

23. Let O be the center of a circle of radius 1. P, Q are the points on the circle such that $\theta = \angle POQ$ is an acute angle and R is a point outside the circle such that OPRQ is a parallelogram. If the area of the part of the parallelogram that is outside the circle is $f(\theta)$, then $\lim_{\theta \to 0} \frac{f(\theta)}{\theta}$ is equal to :

(a) $\frac{3}{\pi}$ (b) $\frac{2}{\pi}$ (c) $\frac{1}{2}$ (d) 1

24. Number of values of $x \in [0, \pi]$ where $f(x) = [4 \sin x - 7]$ is non-derivable is :
[**Note:** $[k]$ denotes the greatest integer less than or equal to k.]

(a) 7 (b) 8 (c) 9 (d) 10

25. Let $P(x) = x^{10} + a_2x^8 + a_3x^6 + a_4x^4 + a_2x^2$ be a polynomial with real coefficients. If $P(1) = 1$ and $P(2) = -5$, then the minimum number of distinct real zeroes of $P(x)$ is

(a) 5 (b) 6 (c) 7 (d) 8

26. Let a sequence of number is as follows

1
3 5
7 9 11
13 15 17 19
21 23 25 27 29
....
....

If t_n is the first term of n^{th} row then $\lim_{n \to \infty} (\sqrt{t_n} - n)$ is equal to :

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) 1 (d) -1

27. Let f be a composite function of x defined by $f(u) = \frac{1}{u^3 - 6u^2 + 11u - 6}$, where $u(x) = \frac{1}{x}$. Then the number of points x where f is discontinuous is :

(a) 4 (b) 3 (c) 2 (d) 1

28. The value of $\lim_{x \to 0^+} \frac{e^{(x^x - 1)} - x^x}{[(x^2)^x - 1]^2}$ is equal to :

(a) 1 (b) $\frac{1}{8}$ (c) $\frac{3}{2}$ (d) $\frac{1}{4}$

29. If $f(x) = \max. \{\sin x, \sin^{-1}(\cos x)\}$, then :

(a) f is differentiable every where
(b) f is continuous every where but not differentiable
(c) f is discontinuous at $x = \frac{n\pi}{2}, n \in I$
(d) f is non-differentiable at $x = \frac{n\pi}{2}, n \in I$

30. Let $f: R \to (0, \infty)$ be such that $f(x) + \dfrac{e^{x+x^2}}{f(x)} \le e^x + e^{x^2}\ \forall\ x > 0$, then $\lim_{x\to 1} f(x)$ is :

(a) 1 (b) $\dfrac{1}{e}$ (c) e (d) $2e$

31. Which of the following statement is true?

(a) The equation $\sin x - x = 0$ has a real root in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

(b) The equation $\tan x - x = 0$ has a real root in $\left(\dfrac{\pi}{6}, \dfrac{\pi}{3}\right)$

(c) If $f(x)$ is a real-valued continuous function in $[0, 2]$ then there exist some $c \in R$ such that $f(x) \ge c$ for all $x \in [0, 2]$

(d) If $g(x)$ is a real-valued function defined on $[3, 5]$ and $g(3) \cdot g(5) < 0$ then there exist some $\alpha \in (3, 5)$ such that $g(\alpha) = 0$

32. If $f(x) = 8 \sin \pi x$ and $g(x) = [f(x)]$ then number of values of x in $[0, 8]$ where $f(x)$ is an integer but $g(x)$ is continuous, is :

(a) 4 (b) 5 (c) 8 (d) 9

[Note : $[y]$ denotes greatest integer function of y.]

33. If $f(x) = (p^2 - 1)\,[\tan^{-1} x] + 4\,(q^2 + 2q - 3)\left\{\dfrac{1}{2+x^2}\right\} + (p + q)\ \text{sgn}\,(x^2 - x + 2)$ is continuous in R and $f(x_1) = f(x_2)\ \forall\ x_1, x_2 \in R$ then largest value of $|p + q|$ is :

(a) 0 (b) 2 (c) 4 (d) 5

[Note : sgn (y), $[y]$ and $\{y\}$ denote signum function, greatest integer function and fractional part function respectively.]

34. The value of $\lim_{x\to 0} \dfrac{\ln(\sec(ex)\sec(e^2x)\ldots\sec(e^{50}x))}{e^2 - e^{2\cos x}}$ is equal to :

(a) $\dfrac{e^{100}-1}{(e^2-1)}$ (b) $\dfrac{e^{100}-1}{2(e^2-1)}$ (c) $\dfrac{2(e^{50}-1)}{e^2-1}$ (d) $\dfrac{e^2\,(e^{100}-1)}{2(e^2-1)}$

35. If $f(x) = \begin{cases} (p^2-1)(\{x\}+2[x]) - 2, & -2 < x \le -1 \\ q\left(\dfrac{e^x + e^{-x}}{2}\right) + |p|(x-1), & -1 < x < 2 \end{cases}$, $p, q \in R$ is continuous in $(-2, 2)$

then $f\left(f\left(f\left(\dfrac{-1}{2}\right)\right)\right)$ is :

(a) -2 (b) -1 (c) 0 (d) not defined

[Note : $[k]$ denotes greatest integer function less than or equal to k and $\{k\}$ denotes fractional part function of k.]

36. If $\Delta_r = \begin{vmatrix} r(r-1) & 2(r+2) & 2015 \\ r & 1 & -6 \\ r^2 & 4r & 0 \end{vmatrix}$ then $\lim_{n\to\infty} \frac{\sum_{r=1}^{n} \Delta_r}{n^4}$ is equal to :

(a) 2 (b) 3 (c) 4 (d) 5

37. If $\lim_{x\to 0} \frac{x+\sin x - x\cos x - \tan x}{x^n}$ exists and is non-zero finite value, then the value of n is :

(a) 3 (b) 4 (c) 5 (d) 6

38. Let $< a_m >$ be the m^{th} term of the sequence, $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}, \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}, \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}$, ...

The value of $\lim_{m\to\infty} (a_m)^{1/m}$ is equals to :

(a) $e+1$ (b) $e-1$ (c) $\frac{1}{e-1}$ (d) $\frac{1}{e+1}$

39. If $\lim_{n\to\infty} [((2009)^{2010})^n + ((2010)^{2009})^n]^{1/n}$ is equal to a^b where $a, b \in N$ then $a-b$ is equal to :

(a) 2009 (b) 1 (c) -1 (d) 0

40. The value of $\lim_{x\to 0^+} \left(\frac{\cot^{-1}(\ln\{x\})}{\pi}\right)^{-\ln\{x\}}$ is equal to :

[**Note:** where { } denotes fractional part function.]

(a) $e^{\frac{1}{\pi}}$ (b) $e^{\frac{-1}{\pi}}$ (c) $e^{\frac{-2}{\pi}}$ (d) $e^{\frac{2}{\pi}}$

41. The value of $\lim_{x\to\pi} \frac{2^{\cot x} + 3^{\cot x} - 5^{1+\cot x} + 2}{(2^{\cot x})^2 + (9^{\cot x})^{1/2} - 5^{\cot x} + 1}$ is :

(a) 5 (b) 2 (c) non-existent (d) -2

42. Let R be the set of real numbers and $f : R \to R$, be a differentiable function such that $|f(x) - f(y)| \le |x-y|^3 \; \forall x, y \in R$. If $f(10) = 100$, the value of $f(20)$ is equal to :

(a) 0 (b) 20 (c) 100 (d) None

43. Let $g(x) = e^{f(x)}$ where $g(x)$ is a differentiable function on $(0, \infty)$ such that $g(x+1) = (x+1)\,g(x)$. Then for $n = 1, 2, 3........ f'(n+1) - f'(1) =$

(a) $-4\left(\frac{1}{1^2}+\frac{1}{2^2}+.....+\frac{1}{n^2}\right)$ (b) $4\left\{\frac{1}{1^2}+\frac{1}{2^2}+....+\frac{1}{n^2}\right\}$

(c) $1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}$ (d) $\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n+1}$

44. Let $f(x), f: R \to R$ be a non-constant continuous function such that $(e^x - 1) f(2x) = (e^{2x} - 1) f(x)$. If $f'(0) = 1$, then $\lim_{x \to 0} \left(\frac{f(x)}{x}\right)^{\frac{1}{x}}$ equal to :

(a) e (b) $e^{1/2}$ (c) e^2 (d) e^{-2}

45. If $f(x)$ is defined $\forall x \in R$ and is discontinuous only at $x = 0$ such that $f^3(x) - 6f^2(x) + 11 f(x) - 3 = 3 \ \forall x \in R$, then the number of such functions is equal to :

(a) 4 (b) 8 (c) 16 (d) 24

46. Let $f(t) = |t| + |t-1| \forall t \in R$ and $g(x) = \begin{cases} \max.(f(t)), & x-1 \le t \le x, \ 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$

Number of points where $g(x)$ is non-derivable in [0, 2], is :

(a) 0 (b) 1 (c) 2 (d) 3

47. Let $f(x) = \begin{cases} \{x^2\}, & -1 \le x < 1 \\ |1 - 2x|, & 1 \le x < 2 \\ (1 - x^2)\,\text{sgn}(x^2 - 3x - 4), & 2 \le x \le 4 \end{cases}$

If m denotes the number of points of discontinuity of $f(x)$ in [– 1, 4] and n denotes the number of points of non-derivability of $f(x)$ in (– 1, 4) then $(m + n)$ equals :

(a) 2 (b) 4 (c) 5 (d) 6

[**Note :** $\{k\}$ and sgn(k) denote fractional part function and signum function of k respectively)]

48. Consider, $f(x) = \min\left\{x^3 - 1, -\frac{1}{4}(|x-2| + |x+2|), 7 - x^3\right\}$ p and q denote number of points where $f(x)$ is discontinuous and non-derivable in [–2, 3] respectively then $p + q$ is :

(a) 0 (b) 1 (c) 3 (d) 4

49. Let $\frac{\pi}{3} < A < \frac{\pi}{2}$ and a function $f(x)$ is defined as :

$$f(x) = \begin{cases} \lim_{n \to \infty} \dfrac{ax^2(\sin A - \sin^3 A) - (5x - b)\,|\sin A - \sin^3 A|^n}{(\sin A - \sin^3 A) - |\sin A - \sin^3 A|^n}, & x \in Q \\ \lim_{n \to \infty} \dfrac{ax^2(\sin A + \sin^3 A) + (5x - b)\,|\sin A + \sin^3 A|^n}{(\sin A + \sin^3 A) + |\sin A + \sin^3 A|^n}, & x \notin Q \end{cases}$$

If $f(x)$ is continuous at $x = 2$ and $x = 3$ then the value of $b - a$, is :

(a) 3 (b) 4 (c) 5 (d) 6

50. Let f be a differentiable function on $(0, \infty)$ and suppose that $\lim_{x\to\infty}(f(x)+f'(x))=L$ where L is a finite quantity, then which of the following must be true?

(a) $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to\infty} f'(x)=L$ (b) $\lim_{x\to\infty} f(x)=\frac{L}{2}$ and $\lim_{x\to\infty} f'(x)=\frac{L}{2}$

(c) $\lim_{x\to\infty} f(x)=L$ and $\lim_{x\to\infty} f'(x)=0$ (d) nothing definite can be said

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 and 2

Let $f(x)=\log_2(x^2+5x)-2\log_2(ax+1)$ where $a>0$

and $g(x)=\begin{cases} 3(\cos^{2m} x)-1, & x<0 \text{ and } m\to\infty \\ -\sin^{2n} x, & x\geq 0 \text{ and } n\to\infty \end{cases}$

1. If $\lim_{x\to\infty}(f(x)+2)=0$, then a equals :

(a) 2 (b) 4 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

2. Which one of the following is true for function $g(x)$?

(a) $g(0^+)\neq g(0)$ (b) $g(0^-)=g(0)$

(c) $g(0^-)\neq g(0^+)$ (d) g is continuous at $x=0$

Paragraph for Question Nos. 3 to 5

Consider an equation $\log_2(\alpha^6-16\alpha^3+66)+\sqrt{4\beta^4-8\beta^2+13}+\left|\left[\frac{\gamma}{3}-2\right]\right|=4$, where α, β, γ are integers and m, n and r are the number of values of α, β and γ respectively which satisfy the above equation.

[**Note :** $[y]$ denotes greatest integer function of y.]

3. The number of ordered triplets (α, β, γ) is :

(a) 2 (b) 3 (c) 6 (d) 9

4. The value of $\lim_{x\to 0}\sum_{i=1}^{r}\left[\frac{\sin(\gamma_i x)}{x}\right]$ is equal to (where $\gamma_1, \gamma_2, \ldots\ldots \gamma_r$ are the values of γ)

(a) 17 (b) 18 (c) 19 (d) 21

5. If $\beta_1 - \beta_2 - \beta_3 - \ldots\ldots - \beta_n = p$ where $\beta_1 > \beta_2 > \ldots\ldots > \beta_n$ and

$\lim_{x\to p}\left[3-2\tan\left(\frac{\pi}{8}x\cdot\beta_1\right)\right]^{\tan\left(\frac{\pi}{8}(\beta_1-\beta_2)x\right)} = e^{\lambda}$ then λ equals :

(a) $-\frac{1}{2}$ (b) -2 (c) 2 (d) 4

Paragraph for Question Nos. 6 and 7

Let $P(x)$ be a polynomial of degree '6' with leading coefficient unity and $p(-x) = p(x)\ \forall x \in R$. Also $(P(1)+3)^2 + P^2(2) + (P(3)-5)^2 = 0$

6. The value of $\lim_{x\to -2}\frac{\sin(P(x))}{(x-2)\tan(x+2)}$ is equal to :

(a) 26 (b) 16 (c) −14 (d) −40

7. The value of $\lim_{x\to\infty}\frac{\left(x^2-\sqrt{\frac{P(x)}{x^2-4}}\right)}{x\tan\frac{1}{x}}$ is equal to :

(a) 10 (b) 5 (c) 3 (d) −4

Paragraph for Question Nos. 8 to 10

Consider $P(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $P(2) = 9$. Let α and β be the roots of the equation $P(x) = 0$.

8. If $\alpha \to \infty$ and $P'(3) = 5$ then $\lim_{x\to\infty}\left(\frac{P(x)}{5(x-1)}\right)^x$ is equal to :

(a) 1 (b) $e^{\frac{1}{5}}$ (c) $e^{\frac{4}{5}}$ (d) $e^{\frac{2}{5}}$

9. If α and β both tends to infinity then $\lim_{x\to 3}\frac{\sqrt{P(x)}-3}{\sin(x-3)}$ is equal to :

(a) 0 (b) 1 (c) 9 (d) non-existent

10. If $\alpha = \beta$, then the value of $\lim_{\alpha\to 0}\left(\lim_{x\to\alpha}\frac{P(x)}{\left[1-\tan^2\left(\frac{\pi}{4}-x+\alpha\right)\right](e^{x-\alpha}-1)}\right)$ is equal to :

(a) $\frac{3}{16}$ (b) $\frac{9}{16}$ (c) $\frac{5}{4e^2}$ (d) $\frac{e^2}{4}$

Paragraph for Question Nos. 11 to 13

Let $P(x) = x^5 - 9x^4 + px^3 - 27x^2 + qx + r$ $(p, q, r \in R)$ be divisible by x^2 and α, β and γ are the positive roots of the equation $\frac{P(x)}{x^2} = 0$.

11. The value of $(p + q + r)$ is equal to :

(a) 9 (b) 27 (c) 81 (d) 108

12. If $\alpha - 1$, $\beta + 3$ and $\gamma + 7$ are the first three terms of a sequence whose sum of first n terms is given by S_n then $\sum_{n=2}^{\infty} \frac{1}{\sqrt{S_n \cdot S_{n-1}}}$ is equal to :

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 2

13. The value of $\lim_{n\to\infty}\left(\frac{1}{p+q+r} + \frac{1}{p^2+q^2+r^2} + \frac{1}{p^3+q^3+r^3} + \ldots + \frac{1}{p^n+q^n+r^n}\right)$ is equal to :

(a) $\frac{1}{26}$ (b) $\frac{1}{27}$ (c) $\frac{25}{26}$ (d) $\frac{26}{27}$

Paragraph for Question Nos. 14 to 16

Let $f(x)$ be a polynomial satisfying $\lim_{x\to\infty} \frac{x^4 f(x)}{x^8 + 1} = 3$, $f(2) = 5$, $f(3) = 10$, $f(-1) = 2$ and $f(-6) = 37$.

14. The value of $f(0)$ is equal to :

(a) 1 (b) 0 (c) 109 (d) 119

15. The value of $\lim_{x\to -6} \frac{f(x) - x^2 - 1}{3(x+6)}$, is equal to :

(a) $-6!$ (b) $6!$ (c) $\frac{6!}{2}$ (d) $-\frac{6!}{2}$

16. The number of points of discontinuity of $g(x) = \frac{1}{x^2 + 1 - f(x)}$ in $\left[\frac{-15}{2}, \frac{5}{2}\right]$, is equal to :

(a) 4 (b) 3 (c) 1 (d) 0

Paragraph for Question Nos. 17 to 19

Consider f, g and h be three real-valued continuous functions on R (the set of all real numbers) defined by

$$f(x) = \begin{cases} x^2 - x + 3, & -\infty < x \le \frac{1}{2} \\ p, & \frac{1}{2} < x \le 1 \\ qx + 4, & 1 < x < \infty \end{cases}, \quad g(x) = \begin{cases} \dfrac{a + b\cos x}{x^2}, & x \ne 0 \\ a - 1, & x = 0 \end{cases} \text{ and}$$

$$h(x) = \begin{cases} x^2 + 1, & x < 1 \\ x^3 + 1, & x \ge 1 \end{cases}$$

17. Which of the following statement(s) is(are) correct?

(a) The value of $(p + q)$ equals $\frac{3}{2}$.

(b) The value of $(p + q)$ equals $\frac{5}{2}$.

(c) The value of $(a + b)$ equals 1.

(d) The value of $(a + b)$ equals 0.

18. Which of the following statement(s) is(are) correct?

(a) Number of real roots of equation $f(x) = 0$ is one.

(b) Number of real roots of equation $h(x) = 0$ is zero.

(c) The value of $g(\pi)$ equals $\frac{4}{\pi^2}$.

(d) The range of function $h(x)$ is R.

19. Which of the following statement(s) is(are) correct?

(a) There exists some $x_0 > 1$ such that $h(x) > f(x)$ is true for all (x_0, ∞).

(b) Both $f(x)$ and $h(x)$ are not injective.

(c) Number of real roots of equation $g(x) = 0$ in $[0, 4\pi]$ are 3.

(d) Range of $f(x)$ is R.

Paragraph for Question Nos. 20 to 22

Let f, g, h be real-valued functions defined on R. Consider $f(x) = -1 + 4\sin^2 x$, $g(x) = 2x - 1$ and $h(x) = 2x + 1$.

20. The range of function $y = \dfrac{h(f(x))}{g(f(x))}$ is equal to :

(a) $\left[\frac{1}{3}, \frac{7}{5}\right]$

(b) $(-\infty, 1) \cup \left[\frac{7}{5}, \infty\right)$

(c) $\left[1, \frac{7}{5}\right]$

(d) $\left(-\infty, \frac{1}{3}\right] \cup \left[\frac{7}{5}, \infty\right)$

21. The value of $\lim\limits_{x\to\frac{1}{2}} \dfrac{f(\sin^{-1} x)}{g(x)}$ is equal to :

(a) 0 (b) 2 (c) 4 (d) 10

22. If $\lim\limits_{x\to\frac{-1}{2}} \dfrac{f(\sin^{-1} kx)-3}{h(x)}$ exists and has value equal to l then $|k| + |l|$ is equal to :

(a) 6 (b) 8 (c) 10 (d) 12

Paragraph for Question Nos. 23 and 24

Let $f(x) = \lim\limits_{n\to\infty} \dfrac{x^2+2(x+1)^{2n}}{(x+1)^{2n+1}+x^2+1}$, $n \in N$ and $g(x) = \tan\left(\dfrac{1}{2}\sin^{-1}\left\{\dfrac{2f(x)}{1+f^2(x)}\right\}\right)$.

23. If $x \in (-2, 0)$ then range of $g(x)$ is

(a) $\left(0, \dfrac{4}{5}\right)$ (b) $\left(0, \dfrac{40}{9}\right)$ (c) $\left(0, \dfrac{5}{4}\right)$ (d) $\left(0, \dfrac{25}{16}\right)$

24. $\lim\limits_{x\to-3} \dfrac{\sin(x+3)\cdot g(x)}{x^2+4x+3}$ is equal to :

(a) 0 (b) -2 (c) $\dfrac{1}{2}$ (d) non-existent

Paragraph for Question Nos. 25 and 26

Consider a circle $x^2 + y^2 = a^2$ and a point P on it in 1st quadrant. Another circle is drawn concentric with the given circle and radius is 'b' more than the x-co-ordinate of point P. This circle intersects positive x-axis at Q and line OP at R (where O is the origin).

25. If angle subtended by arc QR at origin is 'θ' and $\lim\limits_{\theta\to0} \dfrac{\text{Length of arc } QR}{\theta^n} = l$ (where l is a non zero finite quantity, $n \neq 0$) then the value of $(a+n+l)$ for $b = 1$ is equal to :

(a) $\dfrac{5}{2}$ (b) $\dfrac{7}{2}$ (c) $\dfrac{9}{2}$ (d) $\dfrac{11}{2}$

26. If from a point M on the x-axis perpendicular is dropped on the line OP with foot of perpendicular as N and if $\lim\limits_{\theta\to0} \dfrac{MN + \text{Length of arc} QR}{\theta^3} = K$ (where k is a non-zero finite quantity) then co-ordinates of M for $b = 2$ is equal to :

(a) $(-1-a, 0)$ (b) $(-2-a, 0)$ (c) $(-a, 0)$ (d) $(1+a, 0)$

Paragraph for Question Nos. 27 to 29

Let f be a differentiable function satisfying the relation

$$f(x+y) = f(x) + f(y) - 2xy + (e^x - 1)(e^y - 1) \ \forall\, x, y \in R \text{ and } f'(0) = 1.$$

27. $\{f(2)\}$ is equal to :

(a) $e^2 - 5$ (b) $e^2 - 6$ (c) $e^2 - 7$ (d) $e^2 - 8$

[**Note :** $\{y\}$ denotes the fraction part function of y.]

28. The value of $\lim_{x\to 0} \frac{f(2x) + 4x^2 - 2x}{x^2}$ is :

(a) 2 (b) 4 (c) 8 (d) 16

29. Let $g(x) = f(x) + x^2 - 2$. If the equation $|g(|x|)| = k$ has four distinct solutions then the set of values of k is :

(a) $(0, 1)$ (b) $(-2, 2)$ (c) $(0, 3)$ (d) $(0, 2)$

Paragraph for Question Nos. 30 to 32

$$\text{Let } f(x) = \begin{cases} \lim_{n\to\infty} \left(\frac{px}{n} \sum_{r=1}^{n} \frac{[r^2 - e^{-x} + r - 1]}{r(r+1)} \right) + \lambda, & x > 0 \\ q, & x = 0 \\ \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\{r^2 + r + e^x - 1\}}{r(r+1)}, & x < 0 \end{cases}$$

is differentiable in R (the set of all real numbers)

[**Note :** $[y]$ and $\{y\}$ denote greatest integer and fractional part function of y.]

30. The value of $p + q + \lambda$ is equal to :

(a) -2 (b) 2 (c) 1 (d) 3

31. The value of $f'(\ln 2) + f'\left(\ln \frac{1}{2}\right) + f'\left(\ln \frac{3}{2^2}\right) + f'\left(\ln \frac{5}{2^3}\right) + \ldots \infty$ is equal to :

(a) 3 (b) 4 (c) 6 (d) 2

32. If g is the inverse of f then $g'\left(\frac{1}{2}\right)$ is equal to :

(a) $\ln 2$ (b) $-\ln 2$ (c) 2 (d) $\frac{1}{2}$

EXERCISE - 3

More Than One Correct Answers

1. Let $f(x) = \begin{cases} \dfrac{\ln(1+2x)}{x}, & \dfrac{-1}{2} < x < 0 \\ 2\cos x, & x = 0 \\ \dfrac{e^{2x}-1}{x}, & 0 < x < 1 \\ e^2 - 1, & x \geq 1 \end{cases}$

then :

(a) $f(x)$ is continuous at $x = 0$
(b) $f(x)$ is not differentiable at $x = 0$
(c) $f(x)$ is continuous at $x = 1$
(d) $\lim\limits_{x \to 0^+} [f(x)] = 1$

[**Note :** $[k]$ denotes greatest integer less than or equal to k.]

2. Let f be a biquadratic function of x given by $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$, where $A, B, C, D, E \in R$ and $A \neq 0$. If $\lim\limits_{x \to 0}\left(\dfrac{f(-x)}{2x^3}\right)^{\frac{1}{x}} = e^{-3}$, then :

(a) $A + 4B = 0$
(b) $A - 3B = 0$
(c) $f(1) = 8$
(d) $f'(1) = -30$

3. Let $f(x) = \begin{cases} 5x+1, & x \leq 2 \\ \displaystyle\int_0^x (5 + |1-t|)\, dt, & x > 2 \end{cases}$,

then which of the following statement(s) is (are) incorrect?

(a) $f(x)$ is continuous but not differentiable at $x = 2$.
(b) $f(x)$ is not continuous at $x = 2$.
(c) $f(x)$ is differentiable for all $x \in R$.
(d) The right hand derivative of $f(x)$ at $x = 3$ does not exist.

4. A right angled triangle has one leg of length 1 unit, another leg of length x units and hypotenuse is of length y units. The angle opposite to the side of length x units is θ then :

(a) $y^3 - x^3 = \sec^3\theta - \tan^3\theta$
(b) $\lim\limits_{\theta \to \frac{\pi}{2}} (y - x) = 0$
(c) $\lim\limits_{\theta \to \frac{\pi}{2}} (y^2 - x^2) = 1$
(d) $\lim\limits_{\theta \to \frac{\pi}{2}} (y^3 - x^3) = -1$

5. Let $f(x) = \max.(|x|, [x], \cos x), \dfrac{-\pi}{2} < x \leq \dfrac{\pi}{2}$ then :

[**Note :** $[x]$ denotes greatest integer less than or equal to x.]

(a) $f(0) = 1$ (b) $f'(0) = 0$

(c) $f(x) = 1$ has 3 solutions (d) $f\left(\frac{\pi}{2}\right) = 0$

6. Consider, $f(x) = \begin{cases} 2-|x|, & -1 \le x \le 1 \\ |x-2|-x, & 1 < x \le 3 \end{cases}$ and $g(x) = \begin{cases} \sin x - 1, & 0 \le x < \frac{\pi}{2} \\ [x] - \cos(x-2), & \frac{\pi}{2} \le x \le \pi \end{cases}$,

where $[k]$ denotes greatest integer function of k. Identify the correct statement(s).

(a) $\lim\limits_{x\to 1^+} g(f(x)) = -1$

(b) $\lim\limits_{x\to \frac{\pi}{2}^-} g(f(g(x))) = 0$

(c) $\lim\limits_{x\to 2^+} \dfrac{f(g(x))}{f(x)-2} = \dfrac{1}{2}$

(d) $\lim\limits_{x\to 0^+} \dfrac{g(f(x))}{(f(x)-2)^2} = \dfrac{1}{2}$

7. Let $f(x) = \dfrac{1-\cos\{x\}}{(x^4 + ax^3 + bx^2 + cx)^2}$. If $l = \lim\limits_{x\to 1^+} f(x)$, $m = \lim\limits_{x\to 2^+} f(x)$ and $n = \lim\limits_{x\to 3^+} f(x)$,

where l, m and n are non-zero finite then :

(a) $\lim\limits_{x\to 0^+} f(x) = \dfrac{1}{36}$

(b) $a + b + c = -1$

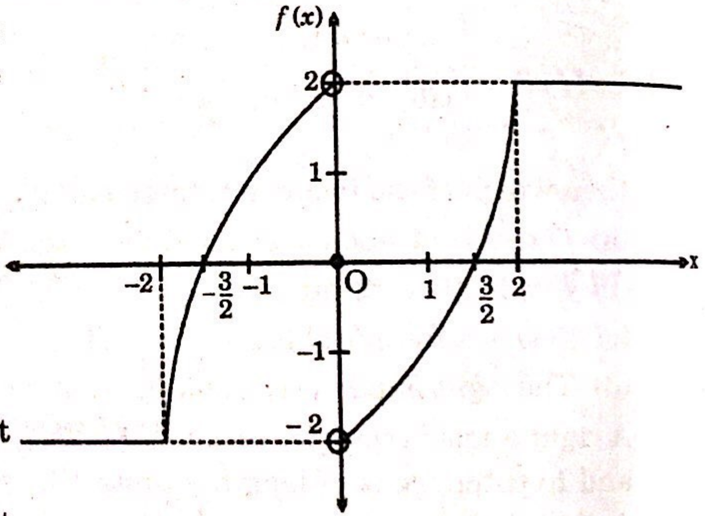
(c) $l + m + n = \dfrac{19}{72}$

(d) $\lim\limits_{x\to 4^+} f(x) = 0$

[Note : $\{y\}$ denotes fractional part of function of 'y'.]

8. The graph of a function $y = f(x)$ is shown in the figure. One more function $y = g(x)$ is defined as :

$$g(x) = \begin{cases} f(x) - 2, & x < 0 \\ f(x), & x = 0 \\ \dfrac{-1}{f(x)} - \dfrac{1}{2}, & 0 < x < \dfrac{3}{2} \\ -f(x) & x \ge \dfrac{3}{2} \end{cases}$$



Identify the correct statement(s)?

[Note : $[k]$ denotes greatest integer less than or equal to k.]

(a) Range of $g(x)$ is $[-4, \infty)$

(b) $\lim\limits_{x\to 0^-} [g(x)] = -1$

(c) $g(x)$ is continuous at $x = 0$

(d) $g(x)$ is discontinuous at $x = \pm\dfrac{3}{2}$

9. Let f be a differentiable function on R satisfying $f(x+y) = f(x)f(y)\ \forall\ x, y \in R$, and $f(1) = 2$. If in a triangle ABC, $a = f(3)$, $b = f(1) + f(3)$, $c = f(3) - f(1)$, then which of the following statement(s) is(are) correct?

[**Note :** All symbols used have usual meaning in triangle ABC.]

(a) Area of triangle is 24.

(b) Radius of circle inscribed in triangle ABC is 2.

(c) Distance between orthocentre and circumcentre of triangle ABC is 5.

(d) The value of $\tan\frac{A}{2}$ equals $\frac{1}{2}$.

10. Let $f(x) = 1 + e^{\ln(\ln x)} \cdot \ln(k^2 + 25)$ and $g(x) = \frac{1}{|x|-1}$. If $\lim_{x\to 1}(f(x))^{g(x)} = k(2\sin^2\alpha + 3\cos\beta + 5)$, $k > 0$ and $\alpha, \beta \in R$, then which of the following is(are) correct?

(a) The value of k is equal to 5.

(b) The value of $\frac{\sin^{10}\alpha + \cos^5\beta}{\sin^2\alpha + \cos\beta}$ is equal to 1.

(c) $\cos^2\beta + \sin^4\alpha = 2$.

(d) $\sin^2\alpha > \cos\beta$.

11. Let f be a non-constant differentiable function satisfying $f(xy) = f(x) f(y)\ \forall\, x, y \in R$ and $f(1 + x) = 1 + x\,(1 + g(x))$ where $\lim_{x\to 0} g(x) = 0$ and $f(1) = 1$, then which of the following statement(s) is(are) correct?

(a) Maximum value of $f(x)$ in [0, 1] is 1.

(b) $\frac{1}{f(x)}$ is unbounded.

(c) $\lim_{x\to 0}(1 + f(x))^{\frac{1}{x}} = \frac{e}{2}$.

(d) The function $\frac{f(x)}{x}$ is a continuous function.

12. Let $f(x) = \lim_{n\to\infty}\frac{\ln(2 + x^2) - x^{2n}\sin(x^2)}{1 + x^{2n}}$. Which of the following statement(s) is(are) correct?

(a) $f(x)$ is discontinuous at two points.

(b) Minimum value of $f(x)$ equals $-\sin 1$.

(c) There exists some $c \in R$ for which $f(c) = 1$.

(d) The equation $f(x) = 0$ has atleast one real root in $(1, \infty)$.

13. Which of the following statement(s) is(are) correct?

(a) If $f(x)$ and $g(x)$ are continuous for every $x \in R$ and fog (x) is defined, then the fog (x) must be continuous for every $x \in R$.

(b) Let $f: R \to R$ be a continuous function such that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$ then $f(x)$ must be bounded.

(c) The equation $\cos\pi x - 3x + 1 = 0$ has no root in (0, 1).

(d) If $f(x)$ is continuous function in $[a, b]$ then there exists some $c \in [a, b]$ such that $f(x) \le f(c)$ for every $x \in [a, b]$.

14. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that

(i) $a_n + b_n + c_n = 2n + 1$
(ii) $a_nb_n + b_nc_n + c_na_n = 2n - 1$
(iii) $a_nb_nc_n = -1$ and
(iv) $a_n > b_n > c_n$

then which of the following is/are correct?

(a) $\lim_{n\to\infty} \frac{a_n}{n} = \frac{1}{2}$ (b) $\lim_{n\to\infty} \frac{a_n}{n} = 2$ (c) $\lim_{n\to-\infty} \frac{a_n}{n} = 0$ (d) $\lim_{n\to-\infty} \frac{a_n}{n} = 2$

15. Let $l_1 = \lim_{x\to\infty} \sqrt{\frac{x - \cos^2 x}{x + \sin x}}$ and $l_2 = \lim_{h\to 0^+} \int_{-1}^{1} \frac{h\,dx}{h^2 + x^2}$. Then

(a) both l_1 and l_2 are less than $\frac{22}{7}$

(b) one of the two limits is rational and other irrational.

(c) $l_2 > l_1$

(d) l_2 is greater than 3 times of l_1.

16. Let f be a differentiable function on R satisfying $f'(t) = e^t(\cos^2 t - \sin 2t)$ and $f(0) = 1$, then which of the following is/are correct?

(a) f is bounded in $x \in (-\infty, 0)$.

(b) Number of solution satisfying the equation $f(t) - e^t = 0$ in $[0, 2\pi]$ is 3.

(c) The value of $\lim_{t\to 0} (f(t))^{\frac{1}{t}} = 1$.

(d) f is neither odd nor even.

17. Let $f: R \to R$ be defined as

$$f(x) = \begin{cases} \lim_{n\to\infty}\left(\frac{[x]}{1+n^2} + \frac{3[x]}{2+n^2} + \frac{5[x]}{3+n^2} + \ldots + \frac{(2n-1)[x]}{n+n^2}\right), & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$$

, where $[y]$ denotes largest integer $\leq y$, then which of the following statement(s) is(are) correct?

(a) $f(x)$ is injective but not surjective.

(b) $f(x)$ is non-derivable at $x = \frac{\pi}{2}$.

(c) $f(x)$ is discontinuous at all integers and continuous at $x = \frac{\pi}{2}$.

(d) $f(x)$ is unbounded function.

18. Which of the following limit vanishes?

(a) $\lim_{x\to 0^+} (x^{x^x} - x^x)$
(b) $\lim_{x\to 0^+} x^2 \ln\sqrt{\frac{1}{x}}$

(c) $\lim_{x\to 0^+} x^{\ln(x+1)}$
(d) $\lim_{x\to 0} \frac{10^x - 2^x - 5^x + 1^x}{x + \tan x}$

19. The possible value(s) of k for which $\lim_{x\to\infty} \dfrac{2x^3-(\tan^{-1}x)^3}{\frac{8}{\pi}x^3\cot^{-1}|kx|+k^2x^6\sin\frac{1}{x^3}-3kx^3}=\dfrac{1}{2}$, is :

(a) 0 (b) -1 (c) 2 (d) 4

20. Consider the function $f: R \to A$ given by $f(x) = x + [x]$ where $[x]$ denotes greatest integer less than or equal to x and A is the range of the function. Then which of the following is/are correct?

(a) $f^{-1}(x)$ exists and is given by $f^{-1}(x) = x - \dfrac{[x]}{2}\ \forall\, x \in A$.

(b) $f^{-1}(x)$ exists and is given by $f^{-1}(x) = x - [x]\ \forall\, x \in A$.

(c) $f(x)$ is a continuous function.

(d) $f^{-1}(x)$ is a continuous function.

21. Let $f(x) = \min.(|e-x|, \pi - |x|)$. Then which of the following statement(s) is (are) correct?

(a) $f(x)$ is many one but not even function.

(b) Range of $f(x)$ is $\left(-\infty, \dfrac{\pi-e}{2}\right]$.

(c) $f(x)$ is continuous and derivable at all integral points.

(d) $f(x)$ is continuous everywhere but non-derivable at exactly two points.

22. Which of the following statements is(are) correct?

[Note : $[x]$ and $\{x\}$ denote greatest integer less than or equal to x and fractional part of x respectively.]

(a) $f(x) = [\ln x] + \sqrt{\{\ln x\}}$, $x > 1$ is continuous at $x = e$.

(b) If $\lim_{x\to -2}\left(\dfrac{3x^2+ax+a+1}{x^2+x-2}\right)$ exists and equals l, then $a + \dfrac{1}{l} = 10$.

(c) $f: [-1, 1] \to [-1, 1]$, $f(x) = x^2 \operatorname{sgn}(x)$ is a bijective function, where sgn x denotes signum function of x.

(d) If f is continuous on $[-1, 1]$, $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.

23. Let $f: R \to R$ be defined as $f(x) = \max.(x, x^2, x^3)$, then :

(a) $f'(x)$ is continuous for all $x \in R$.

(b) f is neither even nor odd function.

(c) $6\int_{-1}^{1} f(x)\,dx = 5$

(d) $f(x)$ is neither injective nor surjective function.

24. In which of the following cases the given equations has atleast one root in the indicated interval?

(a) $x - \cos x = 0$ in $(0, \pi/2)$

(b) $x + \sin x = 1$ in $(0, \pi/6)$

(c) $\frac{a}{x-1} + \frac{b}{x-3} = 0, a, b > 0$ in $(1, 3)$

(d) $f(x) - g(x) = 0$ in (a, b) where f and g are continuous on $[a, b]$ and $f(a) > g(a)$ and $f(b) < g(b)$.

25. Which of the following statements is (are) correct for any function $g : [0, 2] \to [0, 2]$

(a) If g is onto then g must be continuous.

(b) If g is continuous then g must be onto.

(c) g must be bounded.

(d) If g is continuous then g must be bounded.

26. Let $f(x) = \begin{cases} -3+|x|, & -\infty < x < 1 \\ a+|2-x|, & 1 \le x < \infty \end{cases}$ and $g(x) = \begin{cases} 2-|-x|, & -\infty < x < 2 \\ -b+\text{sgn}(x), & 2 \le x < \infty \end{cases}$, where sgn($x$) denotes signum function of x. If $h(x) = f(x) + g(x)$ is discontinuous at exactly one point, then :

(a) $a = -3, b = 0$ (b) $a = -3, b = 1$ (c) $a = 2, b = 1$ (d) $a = 0, b = 1$

27. If $\lim\limits_{x \to 0} \frac{p\cos x + xe^{\frac{1}{x}}}{1 + \sin x + q\cos x \cdot e^{\frac{1}{x}}} = 0$, then which of the following is/are incorrect about p, q?

(a) $p = 0\ q \in R$

(b) $p = 4\ q = 2$

(c) $p = 2\ q \in R$

(d) $p = 0\ q = 2$

28. Let $k \in N$ and $a \in R^+$ $(a \ne 1)$ then $\lim\limits_{n \to \infty} n^k \left(a^{\frac{1}{n}} - 1\right)\left(\sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n+2}}\right)$ is :

(a) 0 if $k \in \{1, 2\}$

(b) $-\ln a$ if $k = 3$

(c) non-existent if $k \ge 4$ and $a \in (0, 1)$

(d) non-existent if $k \ge 4$ and $a > 1$.

29. If $f(x) = \text{sgn}((\sin^2 x - \sin x - 1)(\sin^2 x + \sin x + 1)) = 0$ has exactly 4 points of discontinuity for $x \in (0, n\pi)$, $n \in N$ then the value of n may be equal to :

(a) 2 (b) 3 (c) 4 (d) 5

30. Let $P(x)$ be a polynomial of n degree and $f(x) = \begin{cases} P\left(\frac{1}{x^3}\right)e^{-\frac{1}{x^4}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then

(a) $f(x)$ is discontinuous at $x = 0$
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is non differentiable at $x = 0$
(d) $f'(0) = \lim_{x \to 0} f(x)$

EXERCISE - 4

Match the Columns Type

1. **Note :** $[x]$ and $\{x\}$ denotes the largest integer less than or equal to x and fractional part of x respectively.

Column I	**Column II**
(a) $f(x) = (x+1)\,\lvert x-1 \rvert$	(p) continuous in $(-1, 1)$
(b) $g(x) = \min.(\lvert x \rvert, 1 - \lvert x \rvert)$	(q) differentiable in $(-1, 1)$
(c) $h(x) = \{x\} + 2[x]$	(r) non-differentiable for atleast one point in $(-1, 1)$
	(s) discontinuous at one point in $(-1, 1)$

2. **Column I** — **Column II**

(a) Let f be a real valued differentiable function on R such that $f'(1) = 6$ and $f'(2) = 2$. Then $\lim_{h \to 0} \frac{f(3\cosh + 4\sinh - 2) - f(1)}{f(3e^h - 5\text{sech} + 4) - f(2)}$ is equal to — (p) 4

(b) For $a > 0$, let $f : [-4a, 4a] \to R$ be an even function such that $f(x) = f(4a - x)\ \forall\ x \in [2a, 4a]$ and $\lim_{h \to 0} \frac{f(2a+h) - f(2a)}{h} = 4$, then $\lim_{h \to 0} \frac{f(h-2a) - f(-2a)}{2h}$ is equal to — (q) 5, (r) 3

(c) Suppose f is a differentiable function on R. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. If $f'(1) = e^3$ and $f(0) = f'(0) = 3$, then $\frac{G'(0)}{F'(0)}$ is equal to — (s) 2

(d) Let $f(x) = \max.(\cos x, x, 2x - 1)$ where $x \geq 0$. Then number of points of non-differentiability of $f(x)$, is equal to — (t) 1

3. In the following, [ ] and { } are greatest integer function and fractional part function respectively. Match the functions in column-I with the properties in column-II and indicate your answer by darkening the appropriate bubbles in the 4 × 5 matrix given in *ORS*.

Column I	Column II
(a) $f_1(x) = \left[\frac{4x}{\pi}\right] \text{sgn}\,(x^2 - x + 1)$	(p) discontinuous at more than 3 points but less than 6 points in [–2, 2]
(b) $f_2(x) = \cos^{-1}\left(\text{sgn}\left(\cos\frac{2x-1}{2}\pi\right)\right)$	(q) non derivable at more than 2 points but atmost 5 points in [–2, 2]
(c) $f_3(x) = \text{max.}\,(\{x + 1\}, \{5 - x\})$	(r) range contains atleast one integer but not more than seven and no irrational value in [–2, 2]
(d) $f_4(x) = \sqrt{x^2} + [x]^2$	(s) many one but not even function in [–2, 2]
	(t) neither odd nor periodic in [–2, 2]

4. **Column I** **Column II**

(a) Number of integral values of x satisfying $|x^2 - 9| + |x^2 - 4| = 5$, is — (p) 0

(b) Let $f(x) = \begin{cases} -5, & -\infty < x \le -1 \\ x - 4, & -1 < x \le 6 \\ 2, & 6 < x < \infty \end{cases}$, — (q) 2

then $\lim\limits_{x \to n} \frac{[x]^2 - 13[x] + 42}{(x-7)(x-6)}$ is equal to

[**Note :** Where n is the number of integers in the range of $f(|x|)$, $[x]$ denotes greatest integer less than or equal to x.]

(c) If $\lim\limits_{x \to 0} \frac{a \sin x + bxe^x + 3x^2}{\sin x - 2x + \tan x}$ exists and has value equal to L, — (r) 3

then the value of $\frac{b - L}{a}$, is equal to

(d) Let $P_n = \prod\limits_{k=2}^{n}\left(1 - \frac{1}{{}^{k+1}C_2}\right)$. If $\lim\limits_{n \to \infty} P_n$ can be expressed as — (s) 4

lowest rational in the form $\frac{a}{b}$, then the value of $(b - a)$ is equal to

(t) non-existent

5. Column I **Column II**

(a) Let $f(x)=\begin{cases}\sin x, & x\le 0\\ \tan x, & 0<x<2\pi\\ \cos x, & 2\pi\le x<3\pi\\ -1, & x=3\pi\end{cases}$, then number of points where $f(x)$ is discontinuous in $[0, 3\pi]$, is

(b) Let $f(x)=\begin{cases}2\left(\dfrac{x}{x^2-3x+2}+\dfrac{2}{x^2-5x+6}\right)+k, & x\ne 1,2,3\\ 5, & x=1\\ \lim\limits_{\substack{y\to 0\\ n\to\infty}}\left(\dfrac{\pi}{\sec^{-1}\left(\dfrac{\tan y}{y}\right)^n}\right), & x=2\\ 10, & x=3\end{cases}$

If $f(x)$ is continuous at $x=2$, then the value of k, is

(c) Let $f(x)=x-2$ then the number of points where $\tan^{-1}(|f(|x|)|)$ is non-differentiable, is

(p) 1

(q) 3

(r) 5

(s) 8

6. Column I **Column II**

(a) Let $f(x)=\begin{cases}\sin x, & x\le 0\\ \left[\dfrac{x}{3}\right], & 0<x<2x\\ \cos x, & 2\pi\le x<3\pi\\ -1, & x\ge 3\pi\end{cases}$ then number of points where $f(x)$ is discontinuous in $(-\infty, \infty)$, is equal to

[**Note :** $[k]$ denotes greatest integer less than or equal to k.]

(b) If $\lim\limits_{x\to 0}\dfrac{\ln(1+x+x^2+\dots+x^n)}{nx}$ exists and is equal to $\dfrac{1}{5}$ then the value of n, is equal to

(c) Let $g(x)=|4x^3-x|\cos(\pi x)$ then number of points where $g(x)$ is non-differentiable in $(-\infty, \infty)$, is equal to

(d) Let f be a differentiable function such that $f'(2)=\dfrac{1}{4}$ then $\lim\limits_{h\to 0}\left[\dfrac{f(2+3h^4)-f(2-5h^4)}{h^4}\right]$ is equal to

(p) 1

(q) 2

(r) 3

(s) 4

(t) 5

7. Consider, $f(x) = x - \sin x$, $g(x) = \cos^{-1}(e^{-\frac{x^4}{2}})$ and $h(x) = \tan^{-1}\left(\frac{2x}{1+x^2}\right)$

Column I	Column II
(a) $\lim_{x\to 0} \frac{f(x)-g(x)}{x^2}$ is equal to	(p) -1
(b) $\lim_{x\to 0} \frac{g(x)-xh(x)}{x^2}$ is equal to	(q) $\frac{-2}{3}$
(c) $\lim_{x\to 0} \frac{f(2x)-h(x^3)}{x^3}$ is equal to	(r) $\frac{-1}{2}$
(d) $\lim_{x\to\infty} \frac{\cot^{-1}(h(x))-2g(x)}{\cos^{-1}\left(x\left(1-\cos\frac{1}{x}\right)\right)}$ is equal to	(s) $\frac{1}{2}$

8.

Column I	Column II
(a) If $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$ is L then $\ln(L)$ equals k. The value of $\frac{1}{k^2}$ is equal to	(p) 0
(b) The value of $\lim_{n\to\infty}\left(\frac{n^2}{n-1}\right)^{\tan\frac{1}{\sqrt{n}}}$ is	(q) 1
(c) Number of solution of the equation $8\sin^4 x + 8\cos^4 x = 5$ in the interval $0 < x < 2\pi$ is	(r) 3
(d) Let $f(x)$ be a non-constant polynomial function and $g(x) = \lvert x(x-1)(x-2)f(x)\rvert$. If $g(x)$ is differentiable $\forall x \in R$, then minimum number of distinct roots of $f(x) = 0$ is	(s) 8
	(t) 9

EXERCISE - 5

Integer Answer Type

1. Let $f(x) = \dfrac{\left(\frac{\pi}{2} - \cos^{-1}(1-\{x\}^2)\right)\cdot(\cos^{-1}(1-\{x\}))^2}{2(\{x\}-\{x\}^3)}$. If $f(0^+) = p$ and $f(0^{-1}) = q$, then find the value of $\left(\frac{p\pi}{q}\right)$.

[Note : $\{k\}$ denotes the fractional part of k.]

2. If α and β ($\alpha < \beta$) are the roots of the equation

$$\lim_{t\to\infty} \cos^{-1}\left[\sin\left(\tan^{-1}\left(\frac{\sqrt{tx}}{\sqrt{tx^2 - 3tx + t - 1} - x}\right)\right)\right] = \frac{\pi}{6}$$

then find the value of $(8^{\alpha} + 2^{\beta} - \alpha\beta)$.

3. Let $f(x) = \lim_{n\to\infty} \ln\left(\sqrt{e^{\cos x}\sqrt{e^{3\cos x}\sqrt{e^{5\cos x}\ldots\ldots\sqrt{e^{(2n+1)\cos x}}}}}\right)$. If $g(x) = \left[\frac{1}{3}f(x)\right]$, then find the number of points in $[0, 2\pi]$ where $g(x)$ is discontinuous.

[Note: $[y]$ denotes greatest integer function less than or equal to y.]

4. Let $f(x) = \text{sgn}\,(x^2 - 4x + 4 + k^2)$, $x \in R$. If $f(x)$ is discontinuous at exactly one point then the value of $(\tan^{-1} k + \cos^{-1} k + \text{cosec}^{-1}(2k - 1))$ is equal to $\frac{m\pi}{2}$ where m is a whole number. Find the value of m.

[Note : sgn(k) denotes signum function of k.]

5. If $S_n = \sum_{r=1}^{n} \tan^{-1}\left[\frac{2(2r-1)}{4 + r^2(r^2 - 2r + 1)}\right]$ then find the value of

$$\lim_{n\to\infty} \sum_{n=2}^{n}(\cot(S_{n-1}) - \cot(S_n)).$$

6. If the equation $\left||x - 1| - 6\lim_{t\to\infty}\left(\frac{\sqrt{2t^2 - t - 1} - \sqrt{t^2 - t + 1}}{t\left(\tan\frac{\pi}{8}\right)}\right)\right| = k$ has four distinct solutions then find the number of integral values of k.

7. Let f be a differentiable function satisfying the functional rule $f(xy) = f(x) + f(y) + \frac{x + y - 1}{xy}$ $\forall\, x, y > 0$ and $f'(1) = 2$. Find the value of $[f(e^{100})]$.

Note : $[k]$ denotes the greatest integer less than or equal to k.

8. Let $\lim_{n\to\infty} \frac{1 + cn^2}{(2n + 3 + 2\sin n)^2} = \frac{1}{2}$. If $c \le \alpha \le \beta$ where α and β are the roots of the quadratic equation $x^2 - 2px + p^2 - 1 = 0$, then find the minimum integral value of p.

9. If $f(x) = \lim_{n\to\infty} \frac{\left(1 - \cos\left\{1 - \tan\left(\frac{\pi}{4} - x\right)\right\}\right)(x + 1)^n + \lambda\sin((n - \sqrt{n^2 - 8n})x)}{x^2(x + 1)^n + x}$, $x \ne 0$ is continuous at $x = 0$, then find the value of $(f(0) + 2\lambda)$.

10. If $\sum_{r=1}^{n} \sin^{-1} \alpha_r = \frac{n\pi}{2}$ for any $n \in N$ and $p = \prod_{r=1}^{n} (\alpha_r)^r$.

If $f(x) = \begin{cases} \frac{x^{1/3} - (3-2x)^{1/4}}{x^2 - x}, & x \neq p \\ k, & x = p \end{cases}$ is continuous at $x = p$,

then find the value of $6(k + p)$.

11. Let $l = \lim_{\theta \to 0} \frac{\sin^3\theta - \tan^3\theta}{\theta^5}$ and $m = \lim_{\theta \to \frac{\pi}{4}} \left(\frac{(\sin\theta + \cos\theta)^3 - 2\sqrt{2}}{1 - \sin 2\theta} \right)$. Find the value of $[l^2 + m^2]$(where $[x]$ denotes largest integer less than or equal to x).

12. For $p, n \in N$, let $f(x) = 1 - x^p$ and $g_n(x) = \frac{n}{\frac{1}{f(x)} + \frac{1}{f(2x)} + ... + \frac{1}{f(nx)}}$. Find the value of $\lim_{x \to 0} \frac{1 - g_n(x)}{x^p}$ at $n = 5$ and $p = 3$.

13. Let S_1 and S_2 be two circles of unit radius touching each other externally and the line L is one of the direct common tangent. A circle C_1 with radius r_1 is such that it touches circle S_1, S_2 and the line L. Similarly circles $C_2, C_3, C_4, C_5, ..., C_n$ are the circles of radius $r_2, r_3, r_4, ..., r_n$ where circle C_r touches circle S_1, C_{r-1} and one of the direct common tangents between them. If $\lim_{n \to \infty} \frac{\frac{1}{r_1} + \frac{1}{r_2} + ... + \frac{1}{r_n}}{n^3} = \frac{p}{q}$ where p and q are relatively prime, then find the value of $(p + q)$.

14. Let $L_1 = \lim_{x \to 0} \int_0^x \frac{(1 + \cos t)^2}{x} dt$ and $L_2 = \lim_{x \to \infty} \int_0^x \frac{(1 + \cos t)^2}{x} dt$, then find the value of $2(L_1 + L_2)$.

15. Let $f(x) = x^2 + ax + 3$ and $g(x) = x + b$, where $F(x) = \lim_{n \to \infty} \frac{f(x) + x^{2n} g(x)}{1 + x^{2n}}$. If $F(x)$ is continuous at $x = 1$ and $x = -1$ then find the value of $(a^2 + b^2)$.

16. If $f(x) = \begin{cases} \ln(1 + |x| + (b-2)|x+1|) + a\tan^{-1} x + 2, & -2 < x < 0 \\ b, & x = 0 \\ \frac{(c+2)\sin^{-1}\sqrt{\{x\}} + e^{2x} - e^{\ln\left(4\left\{\frac{x}{2}\right\}\right)} - 1}{x^2}, & 0 < x < 2 \end{cases}$ is derivable in $(-2, 2)$ then find the value of $(9a^2 + b^2 + c^2)$.

[**Note:** $\{y\}$ denotes fractional part of y.]

17. Let $f(x)$ be a function defined in [0, 5] such that $f^2(x) = 1\ \forall\ x \in [0, 5]$ and $f(x)$ is discontinuous only at all integers in [0, 5]. Find total number of possible functions.

18. Let $f:\left[0, \frac{3}{2}\right) \to R$ be a function defined as $f(x) = [3x] - \{2x\}$. Find the number of points of discontinuity of $f(x)$.
[Note : $[y]$ and $\{y\}$ denotes greatest integer less than or equal to y and fractional part of y respectively.]

19. Given a right triangle ABC which is right angled at A with $b < c$. If h_a, w_a and m_a are its altitude bisector and median from the vertex A respectively, then find the value of $\lim\limits_{b \to c} \dfrac{m_a - h_a}{w_a - h_a}$.

20. The sequence $\{a_n\}_{n=1}^{+\infty}$ is defined by $a_1 = 0$ and $a_{n+1} = a_n + 4n + 3$, $n \geq 1$. Find the value of $\lim\limits_{n \to +\infty} \dfrac{\sqrt{a_n} + \sqrt{a_{4n}} + \sqrt{a_{4^2 n}} + \sqrt{a_{4^3 n}} + \ldots + \sqrt{a_{4^{10} n}}}{\sqrt{a_n} + \sqrt{a_{2n}} + \sqrt{a_{2^2 n}} + \sqrt{a_{2^3 n}} + \ldots. + \sqrt{a_{2^{10} n}}}$.

21. The sequence $\langle a_{n-1} \rangle$, $n \in N$ is an arithmetical progression and d is its common difference. If $\lim\limits_{n \to \infty} \left(1 - \dfrac{d^2}{a_1^2}\right)\left(1 - \dfrac{d^2}{a_2^2}\right) \ldots \left(1 - \dfrac{d^2}{a_n^2}\right)$ converges to $\dfrac{1}{4}$ and $a_1 = 8$, then find the value of d.

22. Let the equations $x^3 + 2x^2 + px + q = 0$ and $x^3 + x^2 + px + r = 0$ have two roots in common and the third root of each equation are represented by α and β respectively. If $f(x) = \begin{cases} e^{x \log_{1+x} |\alpha + \beta|}, & -1 < x < 0 \\ a, & x = 0 \\ b\dfrac{\ln(e^{x^2} + \alpha\beta\sqrt{x})}{\tan\sqrt{x}}, & 0 < x < 1 \end{cases}$ is continuous at $x = 0$, then find the value of $2(a + b)$.

23. If the value limit, $\lim\limits_{x \to 0} \dfrac{(1+x)^{1/x} - e + \dfrac{ex}{2}}{x^2} = \dfrac{Ae}{B}$ where A and B are coprime, then find the value of $(A + B)$.

24. If $\lim\limits_{n \to \infty} \left(\dfrac{\left(1 + \dfrac{1}{n}\right)^n}{\left(1 - \dfrac{1}{n}\right)^n} - e^2\right) n^2 = \dfrac{ae^2}{b}$. Find the minimum value of $(a + b)$.

25. Let $f(n) = \lim\limits_{x\to 0} \dfrac{x - n\tan\left(\dfrac{\tan^{-1}x}{n}\right)}{n\sin\left(\dfrac{\tan^{-1}x}{n}\right) - x}$ $(n \in N)$. Find the value of $\sum\limits_{n=1}^{10} \dfrac{2-f(n)}{1+f(n)}$.

26. Let $f(x) = \lim\limits_{n\to\infty} \dfrac{3^n \underbrace{\sin(\sin \ldots\ldots \sin(x))}_{n \text{ times}} + (\sqrt{2}\cos x + 2)^n + 2^n \cos x}{3^n + \sin x(\sqrt{2}\cos x + 2)^n}$, if $l = \lim\limits_{x\to\frac{\pi}{4}^+} f(x)$ and $m = \lim\limits_{x\to\frac{\pi}{4}^-} f(x)$ then find the value of $l^2 + m^2$.

27. If a point $P(x, y)$ lies on the curve $y = f(x)$ such that $\lim\limits_{(x,y)\to(1,2)} \left[\dfrac{\tan^{-1}x + \tan^{-1}\frac{1}{y} - \tan^{-1}3}{(x-1)(y-2)}\right] \sin^{-1}(y-2)$ exists, then find $10\lim\limits_{x\to\frac{1}{3}} \dfrac{f^{-1}(x)}{(3x-1)}$.

28. Let $f : R^+ \to A$ (where A is co-domain of a non empty set) be a function defined as $f(x) = \lim\limits_{a\to\infty} \dfrac{x^{2a} - 3x^a + 2}{x^{2a} + x^a + 1}$. Find the number of elements in A for which f is surjective.

29. Let θ_k $(k = 1, 2, 3, \ldots\ldots, n)$, $\theta_1 < \theta_2 < \ldots\ldots < \theta_n$ be the solution of θ such that $\sqrt{3}\sin n\theta + \cos n\theta = 0$. If $\lim\limits_{n\to\infty} \dfrac{1}{n}\sum\limits_{k=1}^{n} \cos\dfrac{\theta_k}{2} = \dfrac{k}{\pi}$, then find the value of k $(k \in N)$.

30. If the value of $\lim\limits_{x\to 0} \dfrac{(1+3x+2x^2)^{\frac{1}{x}} - (1+3x-2x^2)^{\frac{1}{x}}}{x} = ke^3$, find the value of $12k$.

31. Let $a_1 = 1$ and $a_n = \sin(a_{n-1})\, n > 1, n \in N$. If $\lim\limits_{n\to\infty} \dfrac{2^{2a_n} - 2^{1+a_n}\cdot 3^{a_n} + 3^{2a_n}}{\cos a_n + 1 - e^{a_n} - e^{-a_n}} = -a\ln^2 a$ then, find the value of $3a$.

32. If $f : (0, \infty) \to N$ and

$$f(x) = \left[\frac{x^2+x+1}{x^2+1}\right] + \left[\frac{4x^2+x+2}{2x^2+1}\right] + \left[\frac{9x^2+x+3}{3x^2+1}\right] + \ldots + \left[\frac{n^2x^2+x+n}{nx^2+1}\right], n \in N \text{ then}$$

find the value of $\lim\limits_{n\to\infty}\left[\dfrac{f(x)-n}{(f(x))^2 - \dfrac{n^3(n+2)}{4}}\right]$.

[**Note :** [y] denotes the greatest integer less than or equal to y.]

33. Let $S = \frac{1}{2}\tan\frac{\pi}{4} + \frac{1}{2^2}\tan\frac{\pi}{8} + \frac{1}{2^3}\tan\frac{\pi}{16} + \ldots + \frac{1}{2^n}\tan\left(\frac{\pi}{2^{n+1}}\right)$. If $\lim_{n\to\infty} S = L$; then find the value of $(100\,\pi)\,L$.

(Use may use the fact $\cot x - \tan x = 2\cot 2x$.)

34. Let $\sum_{r=1}^{n} \frac{r^4}{(2r-1)(2r+1)} = \frac{n^3}{A} + \frac{n^2}{B} + \frac{5n}{C} + \frac{f(n)}{D}$ $(A, B, C, D, \in N)$, where $f(n)$ is the ratio of two linear polynomials such that $\lim_{n\to\infty} f(n) = \frac{1}{2}$. Find the value of $(A + B + C + D)$.

35. If $f(x) = \begin{cases} \frac{\tan[x^2]\pi}{ax^2} + ax^3 + b, & 0 < x \le 1 \\ 2\cos\pi x + \tan^{-1} x, & 1 < x \le 2 \end{cases}$ is differentiable in $x \in (0, 2]$. Then $a = \frac{1}{k}$ and $b = \frac{\pi}{4} - \frac{26}{k_2}$. Then find the value of $(k_2 - k_1)$.

36. If $\lim_{x\to 1}\left[\frac{\sin(n\cos^{-1}x)}{\sqrt{1-x^2}} + \frac{1-\cos(n\cos^{-1}x)}{1-x^2}\right]$ (where $n \in N$) exists and is equal to $\frac{3}{2}$, then find the sum of all possible values of n.

37. Let $L = \lim_{n\to\infty} n\left(\sqrt[3]{n^3 + 3n^2 + 2n + 1} + \sqrt{n^2 - 2n + 3} - 2n\right)$. If L can be expressed in the form of $\frac{p}{q}$ $(p, q \in N)$ in the lowest form then find the value of $(p + q)$.

38. If α, β and γ be three distinct real values such that $\frac{\sin\alpha + \sin\beta + \sin\gamma}{\sin(\alpha + \beta + \gamma)} = \frac{\cos\alpha + \cos\beta + \cos\gamma}{\cos(\alpha + \beta + \gamma)} = 2$ and $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = a$, then find the value of $\lim_{x\to a} \frac{\sqrt{x^2 - a^2}}{\sqrt{x-a} + \sqrt{x} - \sqrt{a}}$.

39. Let $f(x) = \begin{cases} \sqrt{(2n+1)x - x^2 - (n^2 + n)} & n \le x < n + \frac{1}{2} \\ n + 1 - x & n + \frac{1}{2} \le x < n + 1 \end{cases}$ $(n \in I)$. Find the number of values of x where $f(x)$ is non-derivable in $(-5, 5)$.

40. Let $X = \{1, 2, 3, 4, 5, 6\}$ and a, c are natural numbers selected from set X with replacement. Let $f(x) = \begin{cases} x^2 + x, & x \le 1 \\ x^2 - ax, & x > 1 \end{cases}$, $g(x) = \begin{cases} x^2 - cx, & x \le 2 \\ x^2 - 4x, & x > 2 \end{cases}$ and $h(x) = f(x) \cdot g(x)$, $x \in R$.

N_1 = Number of function $h(x)$ such that $f(x)$ and $g(x)$ are both discontinuous.

N_2 = Number of function $h(x)$ such that $h(x)$ is continuous at $x = 1$ and discontinuous at $x = 2$ but $f(x)$ and $g(x)$ are both discontinuous.

N_3 = Number of function $h(x)$ such that $h(x)$ is discontinuous at $x = 1$ and continuous at $x = 2$.

Find the value of $(N_1 + N_2 + N_3)$.

41. Find the set of values of a for which the function $f(x) = |x^2 + (a-2)|x| - 2a|$ is non-differentiable at five points.

42. Let $f(x)$ be a continuous function satisfying $f^3(x) - 5f^2(x) + 10f(x) - 12 \le 0$, $f^2(x) - 4f(x) + 3 \ge 0$ and $f^2(x) - 6f(x) + 8 \le 0$ and A is the area bounded by the line $y = x, y = f(x)$ and $x = 0$. Find the value of $10A$.

43. Let $f(x) = \begin{cases} |[x]|, & 0 \le \{x\} < \frac{1}{2} \\ |x|, & \frac{1}{2} \le \{x\} < 1 \end{cases}, \forall x \in \left[\frac{-7}{2}, \frac{7}{2}\right]$. If L is number of point of discontinuity and M is the number of point on non-differentiability of the function $f(x)$, then find the value of $(L + M)$.

[**Note :** $[y]$ and $\{y\}$ denotes greatest integer function less than or equal to y and fractional part of y respectively.]

44. If for a differentiable function $y = f(x), (f'(x) > 0)$ $\lim_{x \to 1} \frac{e^{2f(x)} - 2e^{f(x)+1} + e^2 \cos(x-1)}{\sec^2(x-1) - 1} = \frac{7}{2}e^2$ and area of the triangle formed by the tangent drawn to the curve $y = f(x)$ at $[1, f(1)]$ and co-ordinate axes is Δ, then find the value of $\frac{1}{\Delta}$.

45. Let $f(x) = ax^9 + b\sin x + cx^2 \operatorname{sgn}(x) + \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$ be defined on set of real numbers, $(a > 0, b, c, \in R)$. If $f(-5) = 5, f(-2) = -3$, then find the minimum number of zeros of the equation $f(x) = 0$.

46. If $f(x) = 23 - 2^{-x^2+2x+3} \forall x \in R$ and $g(x) = \left[\frac{f(x)}{\mu}\right]$, where $[k]$ denotes greatest integer function less than or equal to k and $\mu \in N$, then find the sum of all values of μ for which $g(x)$ is discontinuous for at least one real value of x.

47. Let $f : R \to R$ be a function defined as $f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ and $g(x) = f(x-1) + f(x+1) \ \forall x \in R$. Find number of points of non-differentiability of $g(x)$ on R.

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (c)	2. (c)	3. (b)	4. (a)	5. (c)	6. (d)	7. (d)	8. (b)	9. (b)	10. (b)
11. (d)	12. (c)	13. (c)	14. (d)	15. (b)	16. (b)	17. (d)	18. (c)	19. (c)	20. (d)
21. (c)	22. (b)	23. (c)	24. (a)	25. (a)	26. (b)	27. (a)	28. (b)	29. (b)	30. (c)
31. (c)	32. (b)	33. (c)	34. (b)	35. (d)	36. (b)	37. (c)	38. (c)	39. (c)	40. (b)
41. (c)	42. (c)	43. (d)	44. (b)	45. (d)	46. (b)	47. (a)	48. (b)	49. (c)	50. (c)

EXERCISE 2 : Linked Comprehension Type

1. (a)	2. (c)	3. (c)	4. (b)	5. (c)	6. (c)	7. (b)	8. (c)	9. (a)	10. (b)
11. (b)	12. (c)	13. (a)	14. (c)	15. (d)	16. (b)	17. (ad)	18. (abc)	19. (abd)	20. (d)
21. (b)	22. (c)	23. (a)	24. (c)	25. (a)	26. (b)	27. (c)	28. (a)	29. (d)	30. (d)
31. (b)	32. (c)								

EXERCISE 3 : More Than One Correct Answers

1. (a, b, c)	2. (b, d)	3. (b, c, d)	4. (a, b, c)	5. (a, b, c)
6. (b, d)	7. (b, c, d)	8. (a, b, c)	9. (a, b, c, d)	10. (a, b, c)
11. (a, b, d)	12. (a, c, d)	13. (a, b, d)	14. (b, c)	15. (a, b, c, d)
16. (a, b, d)	17. (c, d)	18. (b, d)	19. (a, b, d)	20. (a, d)
21. (a, c)	22. (a, b, c, d)	23. (b, c, d)	24. (a, b, c, d)	25. (c, d)
26. (a, c, d)	27. (a, b, c)	28. (a, b, c, d)	29. (c, d)	30. (b, d)

EXERCISE 4 : Match the Columns Type

1. **(a)** (p) (q), **(b)** (p) (r), **(c)** (r) (s)
2. **(a)** (p), **(b)** (s), **(c)** (r), **(d)** (s)
3. **(a)** (p) (q) (r) (s) (t), **(b)** (p) (q) (s) (t), **(c)** (p) (t), **(d)** (p) (q) (s) (t)
4. **(a)** (s), **(b)** (p), **(c)** (r), **(d)** (q)
5. **(a)** (q), **(b)** (s), **(c)** (q)
6. **(a)** (r), **(b)** (t), **(c)** (p), **(d)** (q)
7. **(a)** (p), **(b)** (p), **(c)** (q), **(d)** (p)
8. **(a)** (t), **(b)** (q), **(c)** (s), **(d)** (r)

EXERCISE 5 : Integer Answer Type

1. 4	2. 9	3. 4	4. 0	5. 2	6. 5	7. 99	8. 3	9. 3	10. 11
11. 6	12. 45	13. 4	14. 11	15. 17	16. 57	17. 162	18. 5	19. 4	20. 683
21. 6	22. 9	23. 35	24. 5	25. 770	26. 2	27. 3	28. 3	29. 2	30. 48
31. 2	32. 2	33. 200	34. 84	35. 6	36. 1	37. 5	38. 2	39. 19	40. 45
41. 2	42. 45	43. 21	44. 4	45. 5	46. 253	47. 5			

❑❑❑

Chapter 4

Methods of Differentiation

KEY CONCEPTS

1. DEFINITION :

If x and $x + h$ belong to the domain of a function f defined by $y = f(x)$, then $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if it exists, is called the **Derivative** of f at x and is denoted by $f'(x)$ or $\frac{dy}{dx}$. We have therefore, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

2. The derivative of a given function f at a point $x = a$ of its domain is defined as :

$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, provided the limit exists and is denoted by $f'(a)$.

Note : Alternatively, we can define $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, provided the limit exists.

3. DERIVATIVE OF $f(x)$ FROM THE FIRST PRINCIPLE /*ab* INITIO METHOD :

If $f(x)$ is a derivable function then, $\lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$.

4. THEOREMS ON DERIVATIVES :

If u and v are derivable function of x, then,

(i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) $\frac{d}{dx}(Ku) = K\frac{du}{dx}$, where K is any constant

(iii) $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} \pm v\frac{du}{dx}$ known as **"Product Rule"**

(iv) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\left(\dfrac{du}{dx}\right) - u\left(\dfrac{dv}{dx}\right)}{v^2}$ where $v \neq 0$ known as **"Quotient Rule"**

(v) If $y = f(u)$ and $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$ **"Chain Rule"**

5. DERIVATIVE OF STANDARDS FUNCTIONS :

(i) $D(x^n) = n x^{n-1}; x \in R, n \in R, x > 0$

(ii) $D(e^x) = e^x$

(iii) $D(a^x) = a^x . \ln a, a > 0$

(iv) $D(\ln x) = \dfrac{1}{x}$

(v) $D(\log_a x) = \dfrac{1}{x}\log_a e$

(vi) $D(\sin x) = \cos x$

(vii) $D(\cos x) = -\sin x$

(viii) $D(\tan x) = \sec^2 x$

(ix) $D(\sec x) = \sec x . \tan x$

(x) $D(\text{cosec } x) = -\text{cosec } x . \cot x$

(xi) $D(\cot x) = -\text{cosec}^2 x$

(xii) $D(\text{constant}) = 0$ where $D = \dfrac{d}{dx}$

6. INVERSE FUNCTIONS AND THEIR DERIVATIVES :

(a) Theorem : If the inverse functions f and g are defined by $y = f(x)$ and $x = g(y)$ and if $f'(x)$ exists and $f'(x) \neq 0$ then $g'(y) = \dfrac{1}{f'(x)}$. This result can also be written as, if $\dfrac{dy}{dx}$ exists and $\dfrac{dy}{dx} \neq 0$, then

$$\frac{dx}{dy} = 1/\left(\frac{dy}{dx}\right) \text{ or } \frac{dy}{dx}\cdot\frac{dx}{dy} = 1 \text{ or } \frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right)\left[\frac{dx}{dy} \neq 0\right]$$

(b) Results :

(i) $D(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(ii) $D(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

(iii) $D(\tan^{-1} x) = \dfrac{1}{1+x^2}, x \in R$

(iv) $D(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2-1}}, |x|>1$

(v) $D(\text{cosec}^{-1} x) = \dfrac{-1}{|x|\sqrt{x^2-1}}, |x|>1$

(vi) $D(\cot^{-1} x) = \dfrac{-1}{1+x^2}, x \in R$

Note : In general if $y = f(u)$ then $\dfrac{dy}{dx} = f'(u) \cdot \dfrac{du}{dx}$.

7. LOGARITHMIC DIFFERENTIATION :

To find the derivative of :

(i) a function which is the product or quotient of a number of functions Or

(ii) a function of the form $[f(x)]^{g(x)}$ where f and g are both derivable, it will be found convenient to take the logarithm of the function first and then differentiate. This is called **Logarithmic Differentiation.**

8. IMPLICIT DIFFERENTIATION :

$\phi(x, y) = 0$

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x and then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit functions, both x and y are present.

9. PARAMETRIC DIFFERENTIATION :

If $y = f(\theta)$ and $x = g(\theta)$ where θ is a parameter, then $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$.

10. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :

Let $y = f(x)$; $z = g(x)$ then $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{f'(x)}{g'(x)}$.

11. DERIVATIVES OF ORDER TWO AND THREE :

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy/dx) or y] and is called the first derivative of y w.r.t. x.

If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w.r.t. x and is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Similarly, the 3[rd] order derivative of y w.r.t. x, if it exists, is defined by $\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$. It is also denoted by $f'''(x)$ or y'''.

12. If $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where $f, g, h, l, m, n, u, v, w$ are differentiable functions of x then $F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$

13. L' HOSPITAL'S RULE :

If $f(x)$ and $g(x)$ are functions of x such that :

(i) $\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x)=\infty=\lim_{x\to a} g(x)$ and

(ii) Both $f(x)$ and $g(x)$ are continuous at $x=a$ and

(iii) Both $f(x)$ and $g(x)$ are differentiable at $x=a$ and

(iv) Both $f'(x)$ and $g'(x)$ are continuous at $x=a$, Then

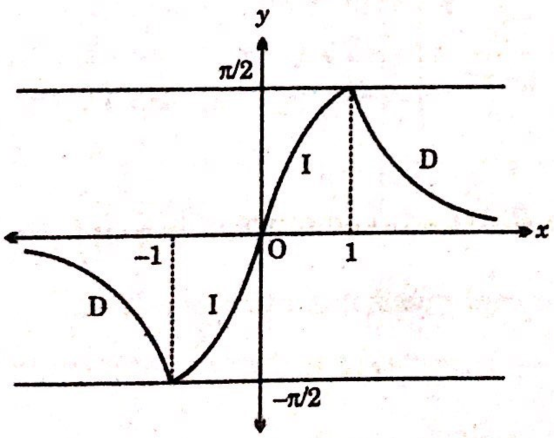
$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}$ and so on till indeterminant form vanishes.

14. ANALYSIS AND GRAPHS OF SOME USEFUL FUNCTIONS :

(i) $y=f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\begin{bmatrix} 2\tan^{-1}x & |x|\le 1 \\ \pi-2\tan^{-1}x & x>1 \\ -(\pi+2\tan^{-1}x) & x<-1 \end{bmatrix}$

Highlights :

(a) Domain is $x \in R$ and range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$



(b) f is continuous for all x but not diff. at $x = 1, -1$

(c) $$\frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & \text{for } |x| < 1 \\ \text{non existent} & \text{for } |x| = 1 \\ -\dfrac{2}{1+x^2} & \text{for } |x| > 1 \end{cases}$$

(d) I in $(-1, 1)$ and D in $(-\infty, -1) \cup (1, \infty)$

(ii) Consider $$y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x & \text{if } x \geq 0 \\ -2\tan^{-1} x & \text{if } x < 0 \end{cases}$$

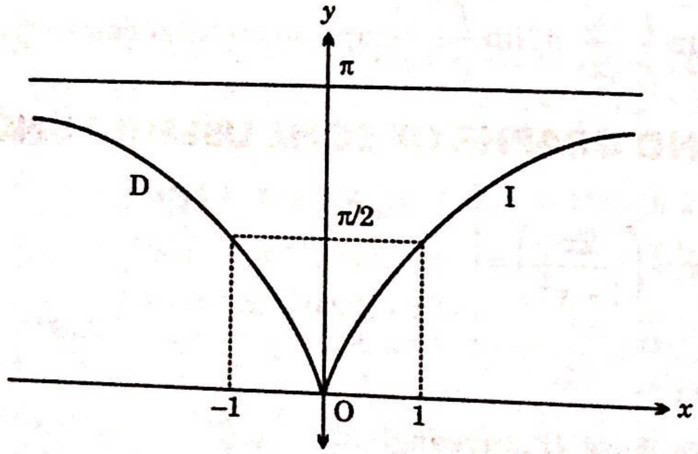
Highlights :

(a) Domain is $x \in R$ and range is $[0, \pi)$

(b) Continuous for all x but not diff. at $x = 0$

(c) $$\frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & \text{for } x > 0 \\ \text{non existent} & \text{for } x = 0 \\ -\dfrac{2}{1+x^2} & \text{for } x < 0 \end{cases}$$

(d) I in $(0, \infty)$ and D in $(-\infty, 0)$



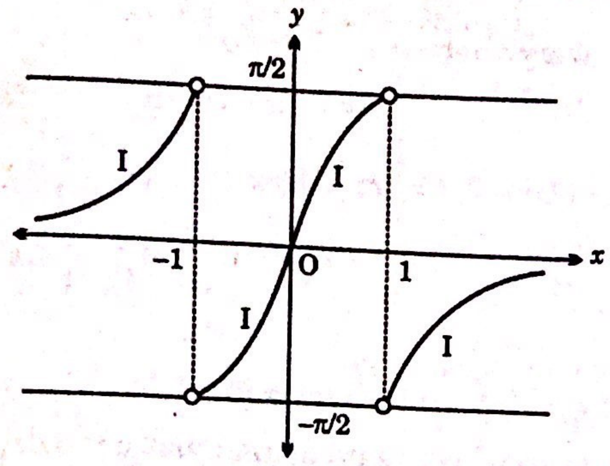
(iii) $$y = f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1} x & |x| < 1 \\ \pi + 2\tan^{-1} x & x < -1 \\ -(\pi - 2\tan^{-1} x) & x > 1 \end{cases}$$

Highlights :

(a) Domain is $R - \{1, -1\}$ and range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(b) f is neither continuous nor diff. at $x = 1, -1$

(c) $$\frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & |x| \neq 1 \\ \text{non existent} & |x| = 1 \end{cases}$$



(d) $I\ \forall\ x$ in its domain

(e) It is bound for all x

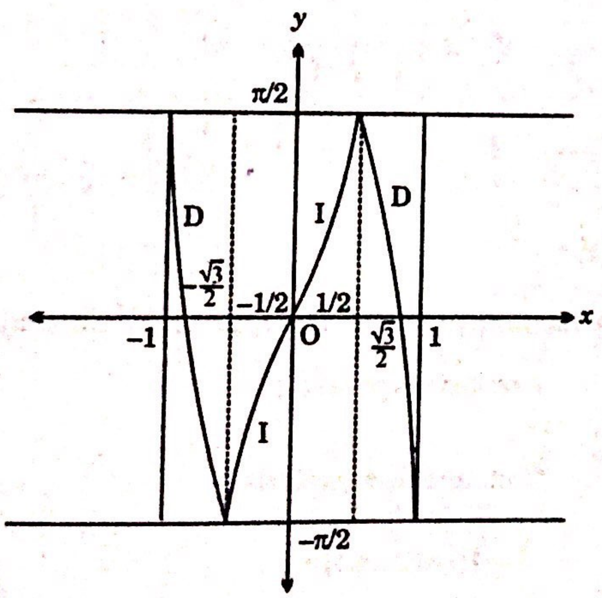
(iv) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if } -1 \le x \le -\frac{1}{2} \\ 3\sin^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

Highlights :

(a) Domain is $x \in [-1, 1]$ and range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) Not derivable at $|x| = \frac{1}{2}$

(c) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$



(d) Continuous everywhere in its domain

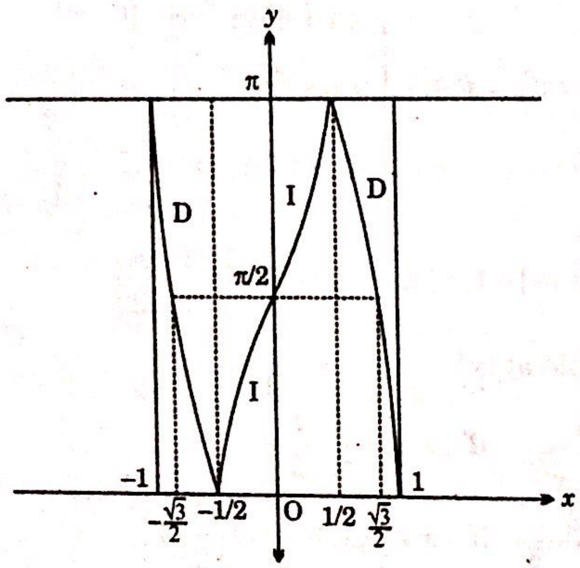
(v) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

Highlights :

(a) Domain is $x \in [-1, 1]$ and range is $[0, \pi]$

(b) Continuous everywhere in its domain but not derivable at $x = \frac{1}{2}, -\frac{1}{2}$

(c) I in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and D in $\left(\frac{1}{2}, 1\right] \cup \left[-1, -\frac{1}{2}\right)$



(d) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

GENERAL NOTE :

Concavity in each case is decided by the sign of 2nd derivative as :

$\frac{d^2y}{dx^2} > 0 \quad \Rightarrow \quad$ Concave upwards

$\frac{d^2y}{dx^2} < 0 \quad \Rightarrow \quad$ Concave downwards

D = Decreasing ; **I = Increasing**

EXERCISE - 1

Only One Correct Answer

1. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is a constant. Then $\frac{d^3[f(x)]}{dx^3}$ at $x = 0$ is :

(a) $6p^3$ (b) $p + p^2$
(c) $p + p^3$ (d) independent of p

2. If $f(x) = (2x - 3\pi)^5 + \frac{4}{3}x + \cos x$ and g is the inverse function of f, then $g'(2\pi)$ is equal to :

(a) $\frac{7}{3}$ (b) $\frac{3}{7}$ (c) $\frac{30\pi^4 + 4}{3}$ (d) $\frac{3}{30\pi^4 + 4}$

3. Given that $\prod_{n=1}^{n} \cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$.

Let $f(x) = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{n=1}^{n} \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right), & x \in (0, \pi) - \left\{\frac{\pi}{2}\right\} \\ \frac{2}{x}, & x = \frac{\pi}{2} \end{cases}$

then which one of the following alternative is **True**?

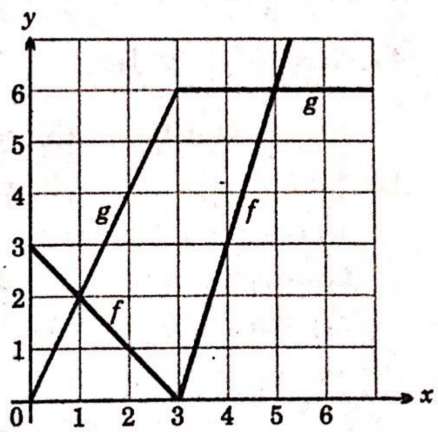
(a) $f(x)$ has non-removable discontinuity of finite type at $x = \frac{\pi}{2}$.

(b) $f(x)$ has missing point discontinuity at $x = \frac{\pi}{2}$.

(c) $f(x)$ is continuous at $x = \frac{\pi}{2}$.

(d) $f(x)$ has non-removable discontinuity of infinite type at $x = \frac{\pi}{2}$.

4. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = \frac{f(x)}{g(x)}$ and $C(x) = f(g(x))$. The value of $[P'(2) - C'(2)]\,Q'(2)$ equals :



(a) 3/2 (b) 3
(c) -3 (d) -6

5. If $y = (A + Bx)\, e^{mx} + (m-1)^{-2} e^x$ then $\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + m^2 y$ is equal to :

(a) e^x (b) e^{mx} (c) e^{-mx} (d) $e^{(1-m)x}$

6. If $y = \frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$, then $\frac{dy}{dx} =$ ....

(a) $2\sin x + \cos x$ (b) $-2\sin x$ (c) $\cos 2x$ (d) $\sin 2x$

7. If $\frac{d}{dx}(f(x)) = g(x)$ and $\frac{d}{dx}(g(x)) = f(x^2)$, then $\frac{d^2}{dx^2}(f(x^3))$ can be expressed in the form $kx^a f(x^b) + px\, g(x^c)$ where $a, b, c, k, p \in N$, then the value of $(k + p + a + b + c)$ equals :

(a) 26 (b) 27 (c) 28 (d) 30

8. Suppose $A = \frac{dy}{dx}$ of $x^2 + y^2 = 0$ at $(\sqrt{2}, \sqrt{2})$, $B = \frac{dy}{dx}$ of $\sin y + \sin x = \sin x \cdot \sin y$ at (π, π) and $C = \frac{dy}{dx}$ of $2e^{xy} + e^x e^y - e^x - e^y = e^{xy+1}$ at $(1, 1)$, then $(A + B + C)$ has the value equal to :

(a) -1 (b) e (c) -3 (d) 0

9. If the function $f(x) = -4e^{\frac{1-x}{2}} + 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$ and $g(x) = f^{-1}(x)$, then the value of $g'\left(-\frac{7}{6}\right)$ equals :

(a) $\frac{1}{5}$ (b) $-\frac{1}{5}$ (c) $\frac{6}{7}$ (d) $-\frac{6}{7}$

10. If $f(x) = \sin^{-1}\{[3x+2] - \{3x + (x - \{2x\})\}\}$, $x \in \left(0, \frac{\pi}{12}\right)$ and $gof(x) = x\ \forall\ x \in \left(0, \frac{\pi}{12}\right)$ then $g'\left(\frac{\pi}{6}\right)$ is equal to :

[Note : $\{y\}$ and $[y]$ denote fractional part function and greatest integer function respectively.]

(a) $\frac{\sqrt{3}}{8}$ (b) $\frac{-1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{-\sqrt{3}}{4}$

11. $\lim_{x \to 0^+} [(x^{x^x}) - x^x]$ is :

(a) equal to 0 (b) equal to 1 (c) equal to -1 (d) non existent

12. If $x_1, x_2, x_3, \ldots, x_{n-1}$ be n zero's of the polynomial $P(x) = x^n + \alpha x + \beta$, where $x_i \neq x_j\ \forall\ i$ and $j = 1, 2, 3, \ldots, (n-1)$. The value of $Q(x) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \ldots (x_1 - x_{n-1})$, is :

(a) $n(n-1)\, x_1^{n-2}$ (b) ${}^nC_2\, x_1^{n-2}$ (c) $(nx_1^{n-1}) + \alpha$ (d) zero

13. Let $f(x)=\begin{cases}\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{x}} & \text{if } |x|<1,\ x\neq 0\\ \dfrac{1}{e^2} & \text{if } x=0\end{cases}$. If $f'(0^+)=p$ and $f'(0^-)=q$, then which one of the following is correct?

(a) p exists but q does not exist
(b) q exists but p does not exist
(c) $p=q=0$
(d) $(p-q)$ is a non zero finite quantity

14. Let f and g be twice differentiable function on R and $f''(5)=8$, $g''(5)=2$ then $\lim\limits_{x\to 5}\left[\dfrac{f(x)-f(5)-(x-5)f'(5)}{g(x)-g(5)-(x-5)g'(5)}\right]$ is equal to :

(a) 0
(b) 1
(c) 2
(d) 4

15. Let $x, y \in R$ satisfying the equation $\cot^{-1}x+\cot^{-1}y+\cot^{-1}(xy)=\dfrac{11\pi}{12}$, then the value of $\dfrac{dy}{dx}$ at $x=1$ is :

(a) $-\left(\dfrac{3+\sqrt{3}}{3}\right)$
(b) $\left(\dfrac{1}{3}+\dfrac{1}{2\sqrt{3}}\right)$
(c) $-\left(\dfrac{5+\sqrt{3}}{3}\right)$
(d) $-\left(\dfrac{1}{3}+\dfrac{1}{2\sqrt{3}}\right)$

16. If $\sin x=\dfrac{2t}{1+t^2}$ and $\cot y=\dfrac{1-t^2}{2t}$, then the value of $\dfrac{d^2x}{dy^2}$, is equal to :

(a) 0
(b) 1
(c) -1
(d) $\dfrac{1}{2}$

17. The value of $\lim\limits_{x\to 0^+}[x^x+(\tan x)^{\operatorname{cosec} x}+(\operatorname{cosec} x)^{\tan x}]$ is equal to :

(a) 1
(b) 2
(c) $2+\dfrac{1}{e}$
(d) $1+\dfrac{1}{e}$

18. Let $f(\alpha)=\cos\left[\cot^{-1}\left(\dfrac{\cos\alpha}{\sqrt{1-\cos 2\alpha}}\right)\right]$ where $\dfrac{\pi}{4}<\alpha<\dfrac{\pi}{2}$, then the value of $\dfrac{df(\alpha)}{d(\cot\alpha)}$, is :

(a) 6
(b) 5
(c) 3
(d) 1

19. The values for A, B and C respectively if $\lim\limits_{x\to 1}\dfrac{Ax^4+Bx^3+1}{(x-1)\sin \pi x}$ exists and is equal to C are :

(a) $-4, 3, \dfrac{6}{\pi}$
(b) $-4, 3, -\dfrac{6}{\pi}$
(c) $3, -4, \dfrac{6}{\pi}$
(d) $3, -4, -\dfrac{6}{\pi}$

20. If a differentiable function $f(x)=e^x+2x$ is given, then $\dfrac{d}{dx}(f^{-1}(x))$ at $x=f(\ln 3)$ is equal to :

(a) $\dfrac{1}{5}$
(b) $\dfrac{3}{7}$
(c) $\dfrac{7}{3}$
(d) 5

21. For the curve $32x^3y^2 = (x + y)^5$, the value of $\frac{d^2y}{dx^2}$ at $P(1, 1)$ is equal to :

(a) 0 (b) 1 (c) –1 (d) $\frac{1}{2}$

22. Let $f:(-2, 2) \to R$ be a differentiable function such that $f(0) = -1$ and $f'(0) = 1$. If $g(x) = (f(2f(x) + 2))^2$ then $g'(0)$ is equal to :

(a) –4 (b) 0 (c) –2 (d) 4

23. Let 'f' be derivable function $\forall x \in R$ such that $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$; $\forall x, y \in R$. If $f'(0) = -1$ and $f(0) = 1$, then :

(a) $2f^{-1}(x) = f(x)$ (b) $f^{-1}(x) = 2f(x)$

(c) $f^{-1}(x) = -f(x)$ (d) $f^{-1}(x) = f(x)$

24. Let $f(x) = \log_3\left(\frac{1-x}{1+x}\right) + \log_3(x + \sqrt{x^2+1})$ then :

(a) the graph of $y = f(x)$ is symmetrical about y-axis.

(b) $f(0) = 1$

(c) $f'(0) = 0$

(d) $f''(0) = 0$

25. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2 and $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$, then $\Delta(x)$ is a polynomial of degree (dashes denote the differentiation).

(a) 2 (b) 3 (c) 0 (d) atmost 3

26. Let function $g(x)$ be differentiable and $g'(x)$ is continuous in $(-\infty, \infty)$ with $g'(2) = 14$, then $\lim_{x\to 0} \frac{g(2+\sin x) - g(2+x\cos x)}{x - \sin x}$ is equal to :

(a) 7 (b) 14 (c) 28 (d) 56

27. If $T(x) = \begin{vmatrix} x^2 - 1 & (x^3+1)\sin(x+1) & (x+1)(e^{x+1}-1) \\ 2^x - \frac{1}{2} & 0 & (x+1)^2\sqrt{x^2+x+1} \\ 0 & x^6 - 1 & 0 \end{vmatrix}$, then $T'''(-1)$ is equal to:

(a) 0 (b) –1 (c) $\frac{1}{2}\ln 2$ (d) does not exist

28. Let $f:[0, 7) \to [1, \infty)$ and $g:[6, \infty) \to [3, \infty)$ be two functions. If $3x - y = 17$ and $y - 2 = 0$ are the tangents to the graph of the functions $f(x)$ and $g(x)$ at $x = 5$ and at $x = 7$ and $h(x) = g^2[x + f(x)]$ then $h'(5)$ is equal to :

(a) 0 (b) 24 (c) 32 (d) $2g(9)g'(9)$

29. If a curve is represented parametrically by the equations $x = f(t)$ and $y = g(t)$ then $\left(\frac{d^2y}{dx^2}\right) \Big/ \left(\frac{d^2x}{dy^2}\right)$ is equal to (where $f'(t) \neq 0, g'(t) \neq 0$) :

(a) 1 (b) $\frac{g'(t)}{f'(t)}$ (c) $-\left[\frac{g'(t)}{f'(t)}\right]^2$ (d) $-\left[\frac{g'(t)}{f'(t)}\right]^3$

30. Let $f(x) = (\log_2 x)^3 + x^3 \ \forall x > 0$, then the derivative of $f^{-1}(x)$ with respect to x at $x = 9$ is :

(a) $\frac{27}{2}$ (b) $\frac{2}{27}$ (c) 27 (d) $\frac{1}{27}$

[Note : You can assume derivative of $\log_a x$ is $\frac{1}{x}$.]

31. If $[f(x)]^3 = 3x^2 - x^3$ and $f''(x) + \frac{nx^2}{[f(x)]^5} = 0$, then the value of '$n$' is :

(a) 1 (b) 2 (c) 3 (d) 4

32. Let $f(x) = ax^3 + bx^2 + cx + 5$. If $|f(x)| \leq |e^x - e^2|$ for all $x \geq 0$ and if the maximum value of $|12a + 4b + c|$ is l, then $[l]$ is equal to :

[Note : $[y]$ denotes greatest integer less than or equal to y.]

(a) 4 (b) 5 (c) 6 (d) 7

33. If $h(x)$ is inverse of $g(x)$ and $g'(x) = \frac{1}{1+x^3}$ then $h''(x)$ is equal to :

(a) $\frac{-3[h(x)]^2}{[1+\{h(x)\}^3]^3}$ (b) $\frac{-3[h(x)]^2}{[1+\{h(x)\}^3]^2}$

(c) $\frac{-3[h(x)]^2}{[1+\{h(x)\}^3]}$ (d) $3h^2(x)[1+h^3(x)]$

34. Let A, B, P be the points the curve $y = \ln x$ with their x coordinates as 1, 2 and t respectively $\lim_{t\to\infty} \cos \angle BAP$ is :

(a) $\sqrt{1+\ln^2 2}$ (b) $\ln 2$ (c) $\frac{1}{\sqrt{1+\ln^2 2}}$ (d) $\frac{1}{1+\ln 2}$

35. Let $f : R \to R$ be defined as $f(x) = x^3 + 2x^2 + 4x + \sin\left(\frac{\pi x}{2}\right)$ and g be the inverse function of f, then $g'(8)$ equals :

(a) $\frac{1}{9}$ (b) 9 (c) $\frac{1}{11}$ (d) 11

36. If $f(x)$, $g(x)$, $h(x)$ are three polynomial functions of degree two and $\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$, then the value of $\lim_{x\to 2} \frac{\phi(x) - \phi(4-x)}{\sin(x-2)}$ is equal to :

(a) 0 (b) 1 (c) 2 (d) 4

37. Let $f : R \to R$ defined by $f(x) = x^3 + 3x + 1$ and g is the inverse of f then the value of $g''(5)$ is equal to :

(a) $\frac{-1}{6}$ (b) $\frac{-1}{36}$ (c) $\frac{-1}{216}$ (d) none of these

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 to 3

Let $f : R \to R$ be a differentiable function such that $f(1) = 1$, $f(2) = 20$, $f(-4) = -4$, $f'(0) = 0$ and $f(x+y) = f(x) + f(y) + 3xy(x+y) + bxy + c(x+y) + 4\ \forall x, y \in R$, where a, b, c are constants.

1. The value of $(b + c)$ is equal to :

(a) 8 (b) 9 (c) 10 (d) 11

2. Number of solutions of the equation $f(x) = x^3 + 4e^x$ is equal to :

(a) 0 (b) 1 (c) 2 (d) 3

3. If $f(x) \geq mx^2 + (5m+1)x + 4m\ \forall x \geq 0$, then the maximum value of m is equal to :

(a) 1 (b) -1 (c) 0 (d) 2

Paragraph for Question Nos. 4 to 6

Consider a function $y = f(x)$. Let the functional rule for $y = f(x)$ is same as the functional rule for the height h (dependent variable) of a triangle ABC from the vertex A to the base BC (where angle A is independent variable). The triangle ABC is inscribed in a circle of radius 6 and the area of the triangle ABC is 12.

4. The least value of $f(x)$ is equal to :

(a) $\frac{1}{2}$ (b) 2 (c) 4 (d) 6

5. If $g(x) = f(\sin^{-1} x)$ then $g'\left(\frac{4}{5}\right)$ is equal to :

(a) $\frac{-25}{4}$ (b) $\frac{-25}{8}$ (c) $\frac{25}{8}$ (d) $\frac{25}{4}$

6. If $h(x) = \sec^{-1}\left[\frac{1}{2}f(x)\right]$ then $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} \dfrac{e^{2h(x)} - 2e^{\left(\frac{\pi}{2}-x\right)} + \sin x}{h(x)\cos x}$ is equal to :

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{3}{2}$ (d) 0

Paragraph for Question Nos. 7 to 9

Consider f and g be two real-valued differentiable functions on R (the set of all real numbers).

Let $f(x) = x^2 + g'(0)\,x + g''(3)$ and $g(y) = f(-2)\,y^2 + y\,f''(y) + f'(y) - 2$

7. $f(1)$ is equal to :

(a) 0 (b) 5 (c) 13 (d) 1

8. Let $h(x) = \begin{cases} f(x), & x \geq 0 \\ g(x), & x < 0 \end{cases}$, then $h(x)$ is :

(a) discontinuous at $x = 0$ (b) continuous but non-derivable at $x = 0$
(c) continuous and derivable at $x = 0$ (d) discontinuous but derivable at $x = 0$

9. If $f(\alpha)g(\beta) \leq 4$, where $\alpha, \beta \in [-3, 3]$ then the number of non-positive ordered pairs (α, β) is equal to :

(a) 0 (b) 1 (c) 2 (d) more than two

Paragraph for Question Nos. 10 and 11

A differentiable function satisfy the relation $\ln(f(x+y)) = \ln(f(x)) \cdot \ln(f(y))$ $\forall\, x, y \in R$, $f(0) \neq 1$, $f'(0) = e$ and g be the inverse of f.

10. The value of $g''(e)$ is equal to :

(a) $\frac{-2}{e^2}$ (b) $\frac{-1}{2e^2}$ (c) $-2e^2$ (d) $\frac{-2}{e^3}$

11. The number of solution(s) of the equation $\ln[\ln f(x)] = g^2[f(x)] - 3g[f(x)] - 5$ is(are) :

(a) 0 (b) 1 (c) 2 (d) infinite

Paragraph for Question Nos. 12 to 14

Let $f, g, f_1, f_2 : R \to R$ be twice differentiable function, and $f(x) \geq 0$, $f'(x) \geq 0$, $g'(x) > 0\ \forall\, x \in R$.

Also $\lim\limits_{x\to\infty} f_1(x) = 5$, $\lim\limits_{x\to\infty} f_2(x) = 12$, $\lim\limits_{x\to\infty} f(x) = \infty$, $\lim\limits_{x\to\infty} g(x) = \infty$

and $\frac{f'(x)}{g'(x)} + f_1(x)\frac{f(x)}{g(x)} = f_2(x)\ \forall\, x > 0$.

(Also $f'(x)$ and $g'(x)$ are continuous).

12. $\lim\limits_{x\to\infty} \frac{g'(x)}{f'(x)}$ equals :

(a) 0 (b) 2 (c) 1 (d) $\frac{1}{2}$

13. $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ equals :

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) 2

14. $\lim_{x\to\infty} \frac{\lambda f'(x)+f''(x)}{\lambda g'(x)+g''(x)}$ equals to ($\lambda > 0$) :

(a) ∞ (b) 0 (c) 2 (d) 1

EXERCISE - 3

More Than One Correct Answers

1. Let $f(x) = \cos^{-1}(2x^2 - 1)$ then :

(a) $f(x)$ is continuous in $[-1, 1]$

(b) $f(x)$ is derivable in $(-1, 1)$

(c) range of $f(x)$ is $(0, \pi)$

(d) derivative of $f(x)$ w.r.t. $\sin^{-1} x$ at $x = \frac{1}{2}$ is 2

2. If the independent variable x is changed to y then the differential equation $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0$ is changed to $x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = k$ where k is equal to :

(a) $\lim_{x\to 0}\left[\frac{2\tan x}{x}\right]$ (b) $\lim_{x\to 0}\left[\frac{2x}{\tan x}\right]$ (c) $\lim_{x\to 0}\left[\frac{\tan x}{x}\right]$ (d) 1

[Note : $[y]$ denotes greatest integer function less than or equal to y.]

3. Let $f : R \to R$ be a differentiable function satisfying $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ $\forall$ $x, y \in R$. If $f(0) = 1$ and $f'(0) = -1$, then which of the following is (are) correct?

(a) $f(|x|)$ is discontinuous at one point

(b) Number of solution of the equation $f(x) = f^{-1}(x)$ is exactly one

(c) $\sum_{r=0}^{10} (f(r))^2 = 286$

(d) $\tan^{-1}(f(x))$ is derivable $\forall\, x \in R$

4. Let a and c be real numbers such that $c > 0$ and $f(x)$ is defined on $[-1, 1]$ as

$$f(x) = \begin{cases} x^a \sin(x^{-c}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Which of the following statement(s) is/are correct ?

(a) $f(x)$ is continuous if and only if $a > 0$
(b) $f'(0)$ exists if and only if $a > 1$
(c) $f'(x)$ is continuous if and only if $a > 1 + c$
(d) $f''(0)$ exists if and only if $a > 2 + c$

5. Let $f: (0, \infty) \to (-\infty, \infty)$ be defined as $f(x) = e^x + \ln x$ and $g = f^{-1}$, then :

(a) $g''(e) = \dfrac{1-e}{(1+e)^3}$ (b) $g''(e) = \dfrac{e-1}{(1+e)^3}$

(c) $g'(e) = e + 1$ (d) $g'(e) = \dfrac{1}{e+1}$

6. Which of the following statements is/are correct?

(a) If $f(x) = x^2 + 10\sin x$, then there exists a real number c such that $f(c) = 1000$

(b) $\dfrac{d}{dx}|x^2 + x| = |2x + 1|\ \forall\, x \in R$

(c) If $y = f(x)$ and $x = g(y)$, where $g = f^{-1}$, then $\dfrac{d^2x}{dy^2} = \dfrac{\left(\dfrac{d^2y}{dx^2}\right)}{\left(\dfrac{dy}{dx}\right)^3}, \dfrac{dy}{dx} \neq 0$

(d) Let $f: (0, 5) \to R - Q$ be a continuous function such that $f(2) = \pi$, then $f(\pi) = \pi$

[**Note:** Q denotes the set of rational numbers.]

7. If $f(\theta) = \tan\left(\sin^{-1}\sqrt{\dfrac{2}{3 + \cos 2\theta}}\right)$, then :

(a) $f'\left(\dfrac{\pi}{4}\right) = \sqrt{2}$ (b) $f\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

(c) $\dfrac{d(f(\theta))}{d(\cos\theta)}$ at $\theta = \dfrac{\pi}{4}$ is $-\sqrt{2}$ (d) $\dfrac{d(f(\theta))}{d(\cos\theta)}$ at $\theta = \dfrac{\pi}{4}$ is -2

8. Let 'f' be a differentiable function satisfying $f(x + y) = f(x) + f(y) + (e^x - 1)(e^y - 1)\ \forall\ x, y \in R$ and $f'(0) = 2$. Identify the correct statement(s) :

(a) $\lim\limits_{x \to 0} \dfrac{f(f(x))}{f(x) - x} = 4$

(b) $\lim\limits_{x \to 0} (f(x) + \cos x)^{\frac{1}{e^x - 1}} = e^2$

(c) Number of solutions of the equation $f(x) = 0$ is 2.

(d) Range of the function $y = f(x)$ is $(-\infty, \infty)$.

9. If $f(x) = \displaystyle\int \frac{x}{2}\, d\left(\frac{x^2 - 1}{x^2}\right)$ and $f(2) = \dfrac{1}{2}$ and $g_r(x) = \underbrace{f[f\{f(.....f(x)....)\}]}_{r-\text{times}}$ i.e. $g_1(x) = f(x)$, $g_2(x) = f[f(x)]$ and so on then identify the correct statement(s).

(a) $\frac{d}{dx}(g_{(3n-2)}(x)) = 1$ whenever exists, $n \in N$

(b) $\frac{d}{dx}(g_{3n}(x)) = 1$ whenever exists, $n \in N$

(c) $\lim\limits_{x\to 1} \frac{\sum\limits_{r=1}^{100}(g_{3r}(x))^r + x - 101}{x-1} = 5050$

(d) Slope of the tangent to the graph of the function $y = g_{80}(x)$ at $x = \frac{1}{2}$ is 4.

EXERCISE - 4

Match the Columns Type

1. Column I contains the function and Column II contains their derivatives at $x = 0$.

Column I	Column II
(a) $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$	(p) 2
(b) $g(x) = \cos^{-1}(2x^2 - 1)$	(q) 3
(c) $h(x) = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$	(r) -2
(d) $k(x) = \tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right]$	(s) non-existent

2. Three functions, f_1, f_2 and f_3 are given as $f_1(x) = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$,

$f_2(x) = x - \cos^{-1}\left(\frac{2x}{1+x^2}\right)$ and $f_3(x) = 4\left\{\frac{\tan^{-1}x}{2}\right] - \{x\}$.

[Note : $\{k\}$ denotes fractional part function of k.]

Column I	Column II
(a) The differential coefficient of $f_1(x)$ with respect to $f_2(x)$ at $x = \tan 1$, is	(p) -1
(b) $\left.\frac{d[f_3(x)]}{d[f_2(x)]}\right\rvert_{x=\tan 1}$ is equal to	(q) 2
(c) $\left.\frac{d}{dx}\left(\int (f_2(x) - f_1(x)\,dx)\right)\right\rvert_{x=\tan 1}$ is equal to	(r) $1 + \sec 2$
(d) The slope of the normal to the curve $y = f_3(x)$ at $x = \sqrt{3}$ is	(s) $\tan 1$

EXERCISE - 5

Integer Answer Type

1. Let $f(x)$ be a differentiable function in $[-1, \infty)$ and $f(0) = 1$ such that $\lim_{t\to x+1} \frac{t^2 f(x+1) - (x+1)^2 f(t)}{f(t) - f(x+1)} = 1$. Find the value of $\lim_{x\to 1} \frac{\ln(f(x)) - \ln 2}{x-1}$.

2. Let the equation $(a-1)x^2 = x(2b+3)$ be satisfied by three distinct values of x, where $a, b \in R$. If $f(x) = (a-1)x^3 + (2b+3)x^2 + 2x + 1$, and $f[g(x)] = 6x - 7$ where $g(x)$ is a linear function then find the value of $g'(2012)$.

3. Let $y = f(x)$ be an infinitely differentiable function on R such that $f(0) \neq 0$ and $\frac{d^n y}{dx^n} \neq 0$ at $x = 0$ for $n = 1, 2, 3, 4$. If $\lim_{x\to 0} \frac{f(4x) + af(3x) + bf(2x) + cf(x) + df(0)}{x^4}$ exists, then find the value of $(25a + 50b + 100c + 500d)$.

4. Let $f(x) = 2\tan^{-1} x$ and $g(x)$ be a differentiable function satisfying $g\left(\frac{x+2y}{3}\right) = \frac{g(x) + 2g(y)}{3}$ $\forall\ x, y \in R$ and $g'(0) = 1$, $g(0) = 2$. Find the number of integers satisfying $f^2(g(x)) - 5f(g(x)) + 4 > 0$ where $x \in (-10, 10)$.

5. If $2x = (y^{1/3} + y^{-1/3})$, then find the value of $\frac{(x^2-1)}{y} \cdot \frac{d^2y}{dx^2} + \frac{x}{y} \cdot \frac{dy}{dx}$.

6. Let $f(x)$ and $g(x)$ be twice differentiable functions satisfying $f(x) = xg(x)$ and $g'(x) = f(x)$, where $g(x) \neq 0\ \forall x \in R$. If $f'(x) = g(x) \cdot h(x)$, then find the number of roots of the equation $h(x) = e^x$.

7. Let f be a function defined implicitly by the equation $\frac{1 - e^{f(x)}}{1 + e^{f(x)}} = x$ and g be the inverse of f. If $g''(\ln 3) - g'(\ln 3) = \frac{p}{q}$ where p and q relative prime numbers then find the value of $(p+q)$.

8. If a differentiable function 'f' satisfies the relation $f(x+y) - f(x-y) = 4xy - 10y$ $\forall$ $x, y \in R$ and $f(1) = 2$, then find the number of points of non-derivability of $g(x) = \left| |f(|x|)| - \frac{1}{4} \right|$.

9. If $y = x^5[\cos(\ln x) + \sin(\ln x)]$, then find the value of $(a+b)$ in the relation $x^2y_2 + axy_1 + by = 0$.

10. If $f(x) = \ln(1 + x^2) + \tan^{-1} x$, $x > 0$ and $g(x) = f^{-1}(x)$ then find the value of $\left|27g''\left(\ln 2e^{\frac{\pi}{4}}\right)\right|$, where $g''(x)$ denotes second derivative of $g(x)$.

11. For the curve $\sin x + \sin y = 1$ lying on the first quadrant, if $\lim\limits_{x\to 0} x^{\alpha} \cdot \dfrac{d^2y}{dx^2}$ exists and has the non-zero value equal to L, find the value of $\left(\dfrac{\alpha}{L}\right)^2$.

12. Let $g(x) = f\left[\dfrac{x}{f(x)}\right]$ where $f(x)$ is a differentiable positive function on $(0, \infty)$ such that $f(1) = f'(1)$. Determine $g'(1)$.

13. If a differentiable function $f(x)$ satisfies a functional rule $f(x) + f(x+2) + f(x+4) = 0\ \forall x \in R$ and $f'(12) = 4$ then find the value of

$$\lim_{x\to 0} \frac{f^2(x+12) - f(x)f(0) - f(x+6)f(18) + f^2(18)}{x\left(\dfrac{\pi}{4} - \tan^{-1}(1-x)\right)}.$$

14. Let $f(x) = \tan[\sin\{\cos^{-1}(\sin(\cos^{-1} x))\} + \cos\{\sin^{-1}(\cos(\sin^{-1} x))\}]$, $x \in (0, 1)$ and $g(x) = \tan x$, $x \in \left(0, \dfrac{\pi}{2}\right)$, then find the value of $\dfrac{d(f(x))}{d(g(x))}$ at $x = \dfrac{\pi}{6}$.

15. Two continuous and differentiable functions $f(x)$ and $g(x)$ are related as $f(x) = \begin{cases} \dfrac{g(x) - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$. If equation of tangent to the curve $y = f(x)$ at $x = 2$ be $y = x + 3$, then find the value of $f(2) + f'(2) + g(2) + g'(2) + g''(2)$.

16. Let $g(x)$ is the only invertible function from $R \to R$ which satisfy the equation $g^3(x) - (x^3 + 2)g^2(x) + (2x^3 + 1)g(x) - x^3 = 0$. Find the value of $g'(8) \cdot (g^{-1})'(8)$.

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (d)	2. (b)	3. (c)	4. (b)	5. (a)	6. (b)	7. (c)	8. (c)	9. (a)	10. (d)
11. (c)	12. (b)	13. (c)	14. (d)	15. (d)	16. (a)	17. (b)	18. (d)	19. (d)	20. (a)
21. (a)	22. (a)	23. (d)	24. (d)	25. (c)	26. (c)	27. (a)	28. (b)	29. (d)	30. (b)
31. (b)	32. (d)	33. (d)	34. (c)	35. (c)	36. (a)	37. (b)			

EXERCISE 2 : Linked Comprehension Type

1. (a)	2. (b)	3. (b)	4. (b)	5. (b)	6. (a)	7. (c)	8. (a)	9. (b)	10. (a)
11. (c)	12. (d)	13. (d)	14. (c)						

EXERCISE 3 : More Than One Correct Answers

1. (a, c)	2. (b, c, d)	3. (c, d)	4. (a, b, c, d)	5. (a, d)
6. (a, d)	7. (a, b, d)	8. (a, b, d)	9. (b, d)	

EXERCISE 4 : Match the Columns Type

1. (a) (r), (b) (s), (c) (s), (d) (q)
2. (a) (r), (b) (p), (c) (s), (d) (q)

EXERCISE 5 : Integer Answer Type

1. 1	2. 3	3. 300	4. 8	5. 9
6. 1	7. 25	8. 9	9. 17	10. 4
11. 18	12. 0	13. 32	14. 6	15. 17
16. 16				

Chapter 5

Indefinite Integration

KEY CONCEPTS

1. DEFINITION :

If f and g are functions of x such that $g'(x) = f(x)$ then the function g is called a **Primitive or Antiderivative or Integral** of $f(x)$ w.r.t. x and is written symbolically as $\int f(x)\,dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x) + c\} = f(x)$, where c is called the **Constant of integration.**

2. STANDARD RESULTS :

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ln a}\,(a > 0) + c$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ln\sec(ax+b) + c$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\ln\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \operatorname{cosec}^2 (ax + b)\, dx = -\frac{1}{a} \cot (ax + b) + c$

(xi) $\int \sec (ax + b) . \tan (ax + b)\, dx = \frac{1}{a} \sec (ax + b) + c$

(xii) $\int \operatorname{cosec} (ax + b) . \cot (ax + b)\, dx = -\frac{1}{a} \operatorname{cosec} (ax + b) + c$

(xiii) $\int \sec x\, dx = \ln (\sec x + \tan x) + c$ ***Or*** $\ln \tan \left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xiv) $\int \operatorname{cosec} x\, dx = \ln (\operatorname{cosec} x - \cot x) + c$ ***Or*** $\ln \tan \frac{x}{2} + c$ ***Or*** $- \ln (\operatorname{cosec} x + \cot x)$

(xv) $\int \sinh x\, dx = \cosh x + c$

(xvi) $\int \cosh x\, dx = \sinh x + c$

(xvii) $\int \operatorname{sech}^2 x\, dx = \tanh x + c$

(xviii) $\int \operatorname{cosech}^2 x\, dx = -\coth x + c$

(xix) $\int \operatorname{sech} x . \tanh x\, dx = - \operatorname{sech} x + c$

(xx) $\int \operatorname{cosech} x . \coth x\, dx = - \operatorname{cosech} x + c$

(xxi) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$

(xxii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(xxiii) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$

(xxiv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln [x + \sqrt{x^2 + a^2}$ ***Or*** $\sinh^{-1} \frac{x}{a} + c$

(xxv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln [x + \sqrt{x^2 - a^2}]$ ***Or*** $\cosh^{-1} \frac{x}{a} + c$

(xxvi) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a + x}{a - x} + c$

(xxvii) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \frac{x - a}{x + a} + c$

(xxviii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

(xxix) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + c$

(xxx) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a} + c$

(xxxi) $\int e^{ax} \cdot \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$

(xxxii) $\int e^{ax} \cdot \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c$

3. TECHNIQUES OF INTEGRATION :

(i) Substitution or change of independent variable. Integral $I = \int f(x)\, dx$ is changed to $\int f(\phi(t)) \, f'(t) \, dt$, by a suitable substitution $x = \phi(t)$ provided the later integral is easier to integrate .

(ii) Integration by Part : $\int u.v \, dx = u \int v \, dx - \int \left[\frac{du}{dx} . \int v \, dx\right] dx$ where u and v are differentiable function.

Note : While using integration by parts, choose u and v such that

(a) $\int v \, dx$ is simple, and

(b) $\int \left[\frac{du}{dx} . \int v \, dx\right] dx$ is simple to integrate.

This is generally obtained, by keeping the order of u and v as per the order of the letters in **ILATE**, where; I – Inverse function, L – Logarithmic function, A – Algebraic function, T – Trigonometric function and E – Exponential function

(iii) Partial fraction, splitting a bigger fraction into smaller fraction by known methods.

4. INTEGRALS OF THE TYPE :

(i) $\int [f(x)]^n \, f'(x) \, dx$ **Or** $\int \frac{f'(x)}{[f(x)]^n} dx$ put $f(x) = t$ and proceed.

(ii) $\int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \sqrt{ax^2 + bx + c} \, dx.$

Express $ax^2 + bx + c$ in the form of perfect square and then apply the standard results .

(iii) $\int \frac{px + q}{ax^2 + bx + c} dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx.$

Express $px + q$ = A (differential co-efficient of denominator) + B.

(iv) $\int e^x [f(x) + f'(x)] \, dx = e^x . f(x) + c$

(v) $\int [f(x) + xf'(x)] \, dx = x f(x) + c$

(vi) $\int \frac{dx}{x(x^n+1)}$ $n \in N$, take x^n common and put $1+x^{-n}=t$.

(vii) $\int \frac{dx}{x^2(x^n+1)^{\frac{(n-1)}{n}}}$ $n \in N$, take x^n common and put $1+x^{-n}=t^n$.

(viii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$ take x^n common as x and put $1+x^{-n}=t$.

(ix) $\int \frac{dx}{a+b\sin^2 x}$ **Or** $\int \frac{dx}{a+b\cos^2 x}$ **Or** $\int \frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$

Multiply Nr and Dr by $\sec^2 x$ and put $\tan x = t$.

(x) $\int \frac{dx}{a+b\sin x}$ **Or** $\int \frac{dx}{a+b\cos x}$ **Or** $\int \frac{dx}{a+b\sin x+c\cos x}$

Hint : Convert sines and cosines into their respective tangents of half the angles, put $\tan\frac{x}{2}=t$

(xi) $\int \frac{a.\cos x+b.\sin x+c}{l.\cos x+m.\sin x+n}dx$. Express $Nr \equiv A\,(Dr)+B\frac{d}{dx}(Dr)+c$ and proceed.

(xii) $\int \frac{x^2+1}{x^4+Kx^2+1}dx$ **Or** $\int \frac{x^2-1}{x^4+Kx^2+1}dx$ where K is any constant.

Hint : Divide Nr and Dr by x^2 and proceed.

(xiii) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ and $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px+q=t^2$.

(xiv) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b=\frac{1}{t}$;

$\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$, put $x=\frac{1}{t}$.

(xv) $\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ **Or** $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x=\alpha\cos^2\theta+\beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ **Or** $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x=\alpha\sec^2\theta-\beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x-\alpha=t^2$ or $x-\beta=t^2$.

EXERCISE - 1

Only One Correct Answer

1. Suppose $\begin{vmatrix} f'(x) & f(x) \\ f''(x) & f'(x) \end{vmatrix} = 0$ where $f(x)$ is continuously differentiable function with $f'(x) \neq 0$ and satisfies $f(0) = 1$ and $f'(0) = 2$ then $f(x)$ is :
   (a) $x^2 + 2x + 1$ (b) $2e^x - 1$ (c) e^{2x} (d) $4e^{x/2} - 3$

2. Let $f(x)$ be a cubic polynomial with leading coefficient unity such that $f(0) = 1$ and all the roots of $f'(x) = 0$ are also roots of $f(x) = 0$. If $\int f(x)\,dx = g(x) + C$, where $g(0) = \frac{1}{4}$ and C is constant of integration, then $g(3) - g(1)$ is equal to :
   (a) 27 (b) 48 (c) 60 (d) 81

3. $\int e^{\tan\theta}(\sec\theta - \sin\theta)\,d\theta$ equals :
   (a) $-e^{\tan\theta}\sin\theta + c$ (b) $e^{\tan\theta}\sin\theta + c$
   (c) $e^{\tan\theta}\sec\theta + c$ (d) $e^{\tan\theta}\cos\theta + c$

4. If $\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x}\,dx = \ln[f(x)] + g(x) + C$ where C is the constant of integration and $f(x)$ is positive, then $f(x) + g(x)$ has the value equal to :
   (a) $e^x + \sin x + 2x$ (b) $e^x + \sin x$ (c) $e^x - \sin x$ (d) $e^x + \sin x + x$

5. If $I = \int \frac{ax^2 + 2bx + c}{(Ax^2 + 2Bx + C)^2}\,dx$ (where $B^2 \neq AC$) is a rational function then which one of the following condition must be necessary?
   (a) $2Bb = Ac + aC$ (b) $Aa + Bb = Cc$
   (c) $Bb = aC + cA$ (d) $\frac{A}{c} + \frac{C}{a} = \frac{2B}{b}$

6. $\int \sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x\,dx$ equals :
   (a) $\frac{\sin 16x}{1024} + c$ (b) $-\frac{\cos 32x}{1024} + c$ (c) $\frac{\cos 32x}{1096} + c$ (d) $-\frac{\cos 32x}{1096} + c$

7. $\int \frac{(2x+1)}{(x^2 + 4x + 1)^{3/2}}\,dx$
   (a) $\frac{x^3}{(x^2 + 4x + 1)^{1/2}} + C$ (b) $\frac{x}{(x^2 + 4x + 1)^{1/2}} + C$
   (c) $\frac{x^2}{(x^2 + 4x + 1)^{1/2}} + C$ (d) $\frac{1}{(x^2 + 4x + 1)^{1/2}} + C$

8. $\int [\sin(101x)\cdot\sin^{99} x]dx$ equals :

(a) $\frac{\sin(100x)(\sin x)^{100}}{100}+C$ (b) $\frac{\cos(100x)(\sin x)^{100}}{100}+C$

(c) $\frac{\cos(100x)(\cos x)^{100}}{100}+C$ (d) $\frac{\sin(100x)(\sin x)^{101}}{101}+C$

9. $\int \frac{\cos\alpha+\cos x+1}{\cos\alpha+\cos x}dx = a\,f(x)+b\,g(x)+c,\ \alpha\in(0,\pi)$

(a) $a=-\cos\alpha,\ b=\sin\alpha,\ f(x)=x+1,\ g(x)=\ln\left|\frac{\tan\frac{x}{2}-\cot\frac{\alpha}{2}}{\tan\frac{x}{2}+\cot\frac{\alpha}{2}}\right|$

(b) $a=\sin\alpha,\ b=\cos^2\alpha,\ f(x)=\sin x,\ g(x)=\tan^{-1}\left(\tan\frac{x}{2}+\tan\frac{\alpha}{2}\right)$

(c) $a=\sin^2\alpha,\ b=-\cos\alpha,\ f(x)=\cos x,\ g(x)=\tan^{-1}\left(\frac{\tan\frac{x}{2}+\cot\frac{\alpha}{2}}{\tan\frac{x}{2}-\cot\frac{\alpha}{2}}\right)$

(d) $a=1,\ b=\operatorname{cosec}\alpha,\ f(x)=x,\ g(x)=\ln\left|\frac{\tan\frac{x}{2}+\cot\frac{\alpha}{2}}{\tan\frac{x}{2}-\cot\frac{\alpha}{2}}\right|$

10. Let f be a polynomial function such that for all real x, $f(x^2+1)=x^4+5x^2+3$, then the primitive of $f(x)$ w.r.t. x, is :

(a) $\frac{x^3}{3}+\frac{3x^2}{2}-x+C$ (b) $\frac{x^3}{3}-\frac{3x^2}{2}+x+C$

(c) $\frac{x^3}{3}-\frac{3x^2}{2}-x+C$ (d) $\frac{x^3}{3}+\frac{3x^2}{2}+x+C$

11. $\int \frac{e^{\tan^{-1}x}}{(1+x^2)}\left[(\sec^{-1}\sqrt{1+x^2})^2+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]dx\ (x>0)$

(a) $e^{\tan^{-1}x}.\tan^{-1}x+C$ (b) $\frac{e^{\tan^{-1}x}.(\tan^{-1}x)^2}{2}+C$

(c) $e^{\tan^{-1}x}.[\sec^{-1}(\sqrt{1+x^2})]^2+C$ (d) $e^{\tan^{-1}x}.[\operatorname{cosec}^{-1}(\sqrt{1+x^2})]^2+C$

12. $\int \frac{(x^2-1)\,dx}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)}=\ln|f(x)|+C$ then $f(x)$ is :

(a) $\ln\left(x+\frac{1}{x}\right)$ (b) $\tan^{-1}\left(x+\frac{1}{x}\right)$ (c) $\cot^{-1}\left(x+\frac{1}{x}\right)$ (d) $\ln\left[\tan^{-1}\left(x+\frac{1}{x}\right)\right]$

13. If $I_n = \int \cot^n x\, dx$, then $I_0 + I_1 + 2(I_2 + I_3 + \ldots\ldots + I_8) + I_9 + I_{10}$ equals to :

(where $u = \cot x$)

(a) $u + \frac{u^2}{2} + \ldots. + \frac{u^9}{9}$

(b) $-\left(u + \frac{u^2}{2} + \ldots. + \frac{u^9}{9}\right)$

(c) $-\left(u + \frac{u^2}{2!} + \ldots. + \frac{u^9}{9!}\right)$

(d) $\frac{u}{2} + \frac{2u^2}{3} + \ldots. + \frac{9u^9}{10}$

14. Let $f : \left[0, \frac{\pi}{2}\right] \to R$ be continuous and satisfy $f'(x) = \frac{1}{1 + \cos x}$ for all $x \in \left(0, \frac{\pi}{2}\right)$.

If $f(0) = 3$ then $f\left(\frac{\pi}{2}\right)$ has the value equal to :

(a) $\frac{13}{4}$ (b) 2 (c) 4 (d) none of these

15. If $\int \frac{(2x+3)\,dx}{x(x+1)(x+2)(x+3)+1} = C - \frac{1}{f(x)}$ where $f(x)$ is of the form of $ax^2 + bx + c$ then

$(a + b + c)$ equals :

(a) 4 (b) 5 (c) 6 (d) none of these

16. Let $\int \frac{dx}{x^{2008} + x} = \frac{1}{p} \ln\left(\frac{x^q}{1 + x^r}\right) + C$, where $p, q, r \in N$ and need not be distinct, then

the value of $(p + q + r)$ equals :

(a) 6024 (b) 6022 (c) 6021 (d) 6020

17. The value of integral $\int e^x \left[\frac{2\tan x}{1 + \tan x} + \cot^2\left(x + \frac{\pi}{4}\right)\right] dx$ is equal to :

(a) $e^x \tan\left(\frac{\pi}{4} - x\right) + C$

(b) $e^x \tan\left(x - \frac{\pi}{4}\right) + C$

(c) $e^x \tan\left(\frac{3\pi}{4} - x\right) + C$

(d) $e^x \tan\left(x - \frac{3\pi}{4}\right) + C$

where C is constant of integration.

18. Let $f(x) = \tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x$, then primitive of $f(x)$ with respect to x is :

(a) $8 \ln(\sin 8x) + C$

(b) $\ln(\sin 8x) + C$

(c) $\ln(\sin x) + C$

(d) $\ln(\sec x) + C$

where C is constant of integration.

19. Let $f : R \to [0, \infty)$ be a differentiable function so that $f'(x)$ is continuous function then $\int \frac{[f(x) - f'(x)]\, e^x}{[e^x + f(x)]^2}\, dx$ is equal to :

(a) $\frac{1}{1 + e^{-x} f(x)} + C$

(b) $\frac{1}{[e^x + f(x)]} + C$

(c) $\dfrac{f'(x)e^x}{e^x+f(x)}+C$ (d) $\dfrac{e^x f(x)}{e^x+f(x)}+C$

where C is constant of integration.

20. If $\int u\dfrac{d^2v}{dx^2}dx = u\dfrac{dv}{dx} - v\dfrac{du}{dx} + w$ then w is equal to :

(a) $\int v\dfrac{d^2u}{dx^2}dx$ (b) $\int u\dfrac{d^2v}{dx^2}dx$ (c) $\int v\left(\dfrac{du}{dx}\right)^2 dx$ (d) $\int u\left(\dfrac{dv}{dx}\right)^2 dx$

21. If $y=y(x)$ and $\left(\dfrac{2+\sin x}{y+1}\right)\dfrac{dy}{dx} = -\cos x$, $y(0)=0$, then $y\left(\dfrac{5\pi}{6}\right)$ equals :

(a) $\dfrac{1}{5}$ (b) $-\dfrac{1}{5}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{2}{5}$

22. For any natural number m, $\int (x^{7m}+x^{2m}+x^m)(2x^{6m}+7x^m+14)^{1/m}dx$ (where $x>0$), equals :

(a) $\dfrac{(7x^{7m}+2x^{2m}+14x^m)^{\frac{m+1}{m}}}{14(m+1)}+C$ (b) $\dfrac{(2x^{7m}+14x^{2m}+7x^m)^{\frac{m+1}{m}}}{14(m+1)}+C$

(c) $\dfrac{(2x^{7m}+7x^{2m}+14x^m)^{\frac{m+1}{m}}}{14(m+1)}+C$ (d) $\dfrac{(7x^{7m}+2x^{2m}+x^m)^{\frac{m+1}{m}}}{14(m+1)}+C$

where C is constant of integration.

23. If the solution of $\dfrac{dy}{dx}=\dfrac{\alpha x+3}{2y+5}$ represents a circle passing through $P(1,1)$ then radius of circle is equal to :

(a) $\dfrac{\sqrt{25}}{2}$ (b) $\dfrac{\sqrt{35}}{2}$ (c) $\dfrac{\sqrt{45}}{2}$ (d) $\dfrac{\sqrt{50}}{2}$

24. $\int \dfrac{\sin x}{\cos^2 x\cdot\sqrt{\cos 2x}}dx$ equals :

(a) $C-\sqrt{1-\tan^2 x}$ (b) $C-\sqrt{1+\tan^2 x}$

(c) $C-\sqrt{1+\cos^2 x}$ (d) $C-\sqrt{1+\sin^2 x}$

where C denotes constant of integration.

25. Let f be a polynomial function such that for all real x, $f(x^2+1)=x^4+5x^2+2$, then $\int f(x)\,dx$ is :

(a) $\dfrac{x^3}{3}+\dfrac{3x^2}{2}-2x+C$ (b) $\dfrac{x^3}{3}+\dfrac{3x^2}{2}+2x+C$

(c) $\dfrac{x^3}{3}-\dfrac{3x^2}{2}-2x+C$ (d) $\dfrac{x^3}{3}-\dfrac{3x^2}{2}+2x+C$

where C is constant of integration.

26. If $\int \frac{dx}{x\sqrt{4-\ln^2 x}} = F(x) + C$, then the value of $F(e) - F(1)$ equals:

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

27. $\int \sin 51x (\sin x)^{49}\, dx$ equals :

(a) $\frac{\sin 50x(\sin x)^{50}}{50} + C$ (b) $\frac{\cos 50x(\sin x)^{50}}{50} + C$

(c) $\frac{\cos 50x(\cos x)^{50}}{50} + C$ (d) $\frac{\sin 50x(\sin x)^{51}}{51} + C$

28. $\int P(x) \cdot e^{kx} dx = Q(x)e^{4x} + C$, where $P(x)$ is polynomial of degree n and $Q(x)$ is polynomial of degree 7. Then the value of $n + 7 + k + \lim_{x\to\infty} \frac{P(x)}{Q(x)}$ is :

(a) 18 (b) 19

(c) 20 (d) 22

29. Let $f(x)$ be a cubic polynomial with leading coefficient unity such that $f(0) = 1$ and all the roots of $f'(x) = 0$ are also roots of $f(x) = 0$. If $\int f(x)\, dx = g(x) + C$, where $g(0) = \frac{1}{4}$ and C is constant of integration, then $g(3) - g(1)$ is equal to :

(a) 27 (b) 48

(c) 60 (d) 81

30. If $\int \frac{dx}{x + x^{2011}} = f(x) + C_1$ and $\int \frac{x^{2009}}{1 + x^{2010}} dx = g(x) + C_2$ (where C_1 and C_2 are constants of integration). Let $h(x) = f(x) + g(x)$. If $h(1) = 0$ then $h(e)$ is equal to :

(a) 0 (b) 1 (c) e (d) 2

31. Let $A = 2^{\left(\frac{\log_{10}\left(\frac{100}{x}-1\right)}{-\log_{10} 2}\right)}$ then $\int \ln 10 \cdot \log_{10}\left(\frac{1}{A}\right) dx$ is equal to :

(a) $(x - 100)\ln(100 - x) - x\ln x + C$

(b) $(x - 100)\ln(100 - x) + x - x\ln x + C$

(c) $(100 - x)\ln(100 - x) - x\ln x + C$

(d) $(100 - x)\ln(100 - x) + x\ln x + C$

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 to 3

Let $f(x)$ be a polynomial function of degree 2 satisfying

$$\int \frac{f(x)}{x^3-1}dx = \ln\left|\frac{x^2+x+1}{x-1}\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C,$$

where C is indefinite integration constant.

1. The value of $f(1)$ is equal to :

(a) 1 (b) 2 (c) -1 (d) -3

2. Let $\int \frac{1-6\operatorname{cosec} x}{6+f(\sin x)} d(\sin x) = g(x) + K$, where $g(x)$ contains no constant term.

Then $\lim\limits_{t\to\frac{\pi}{2}} g(t)$ is equal to (where K is indefinite integration constant.)

(a) ln 1 (b) ln 2 (c) ln 3 (d) ln 4

3. Let $\int \frac{5+f(\sin x)+f(\cos x)}{\sin x+\cos x}dx = h(x)+\lambda$, where $h(1) = -1$. The value of $\tan^{-1}[h(2)]+$ $\tan^{-1}[h(3)]$ is equal to (where λ is indefinite integration constant.)

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $-\frac{3\pi}{4}$

Paragraph for Question Nos. 4 and 5

Consider $\phi(a, b, t) = a^4 - 5a^2 + b^2 + 5t^2 - 4bt - 2t + \frac{33}{4}$ where $a, b, t \in R$. Given that $f(t)$ and $g(b)$ are the minimum values of $\phi(a, b, t)$.

4. $\int \frac{dx}{f(x)}$ is :

(a) $\tan^{-1}(x-1)+C$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C$

(c) $\ln[x-1+\sqrt{x^2-2x+2})+C$ (d) $\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$

[Note: Where C is the constant of integration]

5. If $\int e^x g(x)\,dx = e^x(Ax^2+Bx+C)+D$, where D is constant of integration, then $(A+B+C)$ is equal to :

(a) 2 (b) $\frac{14}{5}$ (c) 5 (d) $\frac{19}{5}$

Paragraph for Question Nos. 6 to 8

Let a differentiable function '*f*' satisfies the functional rule

$f(xy) = f(x) + f(y) + xy - x - y \ \forall\ x, y > 0$ and $f'(1) = 4$.

6. If $f(x_0) = 0$, then x_0 lies in the interval :

(a) $(0, 1)$ (b) $(1, e)$ (c) (e, e^2) (d) (e^2, e^3)

7. $\int \frac{f(x)}{x} dx$ is equal to :

(a) $3(\ln x)^2 + x + c$ (b) $3 \ln x + x + c$

(c) $\frac{3}{2} \ln x + x + c$ (d) $\frac{3}{2}(\ln x)^2 + x + c$

8. If $\int e^{f(x)} dx = e^x(ax^3 + bx^2 + cx + d) + \lambda$, then the value of $(a + b + c + d)$ is equal to :

(a) -1 (b) -2 (c) 3 (d) 6

EXERCISE - 3

More Than One Correct Answers

1. $\int \frac{x\,dx}{x^4 + x^2 + 1}$ equals

(a) $\frac{2}{3} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C$

(b) $\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x - 1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right)\right] + C$

(c) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C$

(d) $\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x - 1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right)\right] + C$

where C is an arbitrary constant.

2. If $\int \frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x} dx = pf(x) + qg(x) + c$ where 'c' is a constant of integration, then :

(a) $p = 1; q = \frac{1}{\sqrt{3}}; f(x) = x; g(x) = \ln\left|\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right|$

(b) $p = 1; q = -\frac{1}{\sqrt{3}}; f(x) = x; g(x) = \ln\left|\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right|$

(c) $p=1; q=-\frac{2}{\sqrt{3}}; f(x)=x; g(x)=\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|$

(d) $p=1; q=-\frac{1}{\sqrt{3}}; f(x)=x; g(x)=\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|$

3. $\int\sqrt{1+\cos x}\,dx$ equals :

(a) $2\sin^{-1}\sqrt{\sin x}+c$

(b) $\sqrt{2}\cos^{-1}\sqrt{\cos x}+c$

(c) $c-2\sin^{-1}(1-2\sin x)$

(d) $\cos^{-1}(1-2\sin x)+c$

4. Suppose $J=\int\frac{\sin^2 x+\sin x}{1+\sin x+\cos x}dx$ and $K=\int\frac{\cos^2 x+\cos x}{1+\sin x+\cos x}dx$. If C is an arbitrary constant of integration then which of the following is/are correct?

(a) $J=\frac{1}{2}(x-\sin x+\cos x)+C$

(b) $J=K-(\sin x+\cos x)+C$

(c) $J=x-K+C$

(d) $K=\frac{1}{2}(x-\sin x+\cos x)+C$

EXERCISE - 4

Match the Columns Type

1. Let $I=\int\frac{e^x}{e^{4x}+1}dx$ and $J=\int\frac{e^{-x}}{e^{-4x}+1}dx$. Then for any arbitrary constant C, match the following

Column I	Column II
(a) I	(p) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+C$
(b) $J+I$	(q) $\frac{1}{2\sqrt{2}}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)+C$
(c) $J-I$	(r) $\frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)-\frac{1}{2}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)\right]+C$
	(s) $\frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+\frac{1}{2}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)\right]+C$

EXERCISE - 5

Integer Answer Type

1. Let $\frac{d}{dx}(x^2y) = x - 1$ where $x \neq 0$ and $y = 0$ when $x = 1$. Find the smallest natural x for which $\frac{dy}{dx}$ is positive.

2. Suppose $\int \frac{1-7\cos^2 x}{\sin^7 x \cos^2 x} dx = \frac{g(x)}{\sin^7 x} + C$, where C is an arbitrary constant of integration. Then find the value of $g'(0) + g''\left(\frac{\pi}{4}\right)$.

3. If $\int \left(\frac{\sin x + \sin 3x + \sin 5x + \sin 7x + \sin 9x + \sin 11x + \sin 13x + \sin 15x}{\cos x + \cos 3x + \cos 5x + \cos 7x + \cos 9x + \cos 11x + \cos 13x + \cos 15x}\right) dx$ equals $\frac{\ln(\sec mx)}{n}$ where $m, n \in N$, find $(m + n)$.

4. Let $\int \frac{(1+x^4)\, dx}{(1-x^4)^{\frac{3}{2}}} = f(x) + C_1$ where $f(0) = 0$ and $\int f(x)\, dx = g(x) + C_2$ with $g(0) = 0$. If $g\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{k}$. Find k.

5. Let $f'(x^2) = \frac{1}{x}$ for $x > 0$, $f(1) = 1$ and $g'(\sin^2 x - 1) = \cos^2 x + p\ \forall\ x \in R$, $g(-1) = 0$. If $h(x) = \begin{cases} f(x), & x > 0 \\ g(x), & -1 \le x \le 0 \end{cases}$ is a continuous function, then find the absolute value of $2p$.

6. If $F(x) = \int \frac{(1+x)[(1-x+x^2)(1+x+x^2)+x^2]}{1+2x+3x^2+4x^3+3x^4+2x^5+x^6} dx$ then find the value of $[F(99) - F(3)]$.

[Note : $[k]$ denotes greatest integer less than or equal to k.]

7. Let $\int \sec^{-1}[-\sin^2 x] dx = f(x) + C$ where $[y]$ denotes largest integer $\le y$, then find the value of $\left[f\left(\frac{8}{\pi x}\right)\right]''$ at $x = 2$.

8. If $\int (x^{2010} + x^{804} + x^{402})(2x^{1608} + 5x^{402} + 10)^{\frac{1}{402}} dx$

$= \frac{1}{10a}(2x^{2010} + 5x^{804} + 10x^{402})^{\frac{a}{402}}$, then find the value of a.

9. If $\int \frac{(x-1)\,dx}{(x+x\sqrt{x}+\sqrt{x})\sqrt{\sqrt{x}(x+1)}} = 4\tan^{-1}[g(x)] + C$, where C is an arbitrary constant of integration. Find $g^2(1)$.

10. If $\int (\cot 2x \cot 3x - \tan 2x \tan 7x) \tan 5x = a \ln(\tan 2x) + b \ln(\sin 3x) + c \ln(\sec 5x) + d \ln(\cos 7x) + C$ and $a, b, c, d \in Q$ and C is the constant of integration. If $(a+b+c+d)$ can be expressed as $\frac{m}{n}$ in the lowest form, find $(m+n)$.

11. Suppose $\int \frac{1-7\cos^2 x}{\sin^7 x \cos^2 x}\,dx = \frac{g(x)}{\sin^7 x} + C$, where C is an arbitrary constant of integration. Then find the value of $g'(0) + g''\left(\frac{\pi}{4}\right)$.

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (c) 2. (c) 3. (d) 4. (b) 5. (a) 6. (b) 7. (b) 8. (a) 9. (d) 10. (a)
11. (c) 12. (b) 13. (b) 14. (c) 15. (b) 16. (c) 17. (b) 18. (c) 19. (a) 20. (a)
21. (b) 22. (c) 23. (d) 24. (a) 25. (a) 26. (b) 27. (a) 28. (d) 29. (c) 30. (b)
31. (a)

EXERCISE 2 : Linked Comprehension Type

1. (d) 2. (c) 3. (d) 4. (a) 5. (a) 6. (a) 7. (d) 8. (b)

EXERCISE 3 : More Than One Correct Answers

1. (b, c) 2. (a, d) 3. (a, d) 4. (b, c)

EXERCISE 4 : Match the Columns Type

1. (a) (r), (b) (p), (c) (q)

EXERCISE 5 : Integer Answer Type

1. 2 2. 5 3. 16 4. 12 5. 3
6. 3 7. 2 8. 403 9. 2 10. 499
11. 5

Chapter 6

Definite Integration

KEY CONCEPTS

1. $\int_a^b f(x)\,dx = F(b) - F(a)$ where $\int f(x)\,dx = F(x) + c$

Note : If $\int_a^b f(x)\,dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b).

2. PROPERTIES OF DEFINITE INTEGRAL :

P–1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ provided f is same

P–2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P–3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval $[a, b]$. This property to be used when f is piecewise continuous in (a, b).

P–4 $\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is an odd function *i.e.* $f(x) = -f(-x)$.

$= 2\int_0^a f(x)\,dx$ if $f(x)$ is an even function *i.e.* $f(x) = f(-x)$.

P–5 $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$, In particular $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$

P-6 $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x) = 0$

if $f(2a-x) = -f(x)$

P-7 $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$; where '$a$' is the period of the function *i.e.* $f(a+x) = f(x)$

P-8 $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$ where $f(x)$ is periodic with period T and $n \in I$.

P-9 $\int_{ma}^{na} f(x)\,dx = (n-m)\int_0^a f(x)\,dx$ if $f(x)$ is periodic with period 'a'.

P-10 If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

P-11 $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

P-12 If $f(x) \ge 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \ge 0$.

3. WALLI'S FORMULA :

$$\int_0^{\pi/2} \sin^n x.\cos^m x\,dx = \frac{[(n-1)(n-3)(n-5)\ldots.1 \text{ or } 2][(m-1)(m-3)\ldots 1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)\ldots.1 \text{ or } 2}K$$

Where $K = \frac{\pi}{2}$ if both m and n are even $(m, n \in N)$;

$= 1$ otherwise

4. DERIVATIVE OF ANTIDERIVATIVE FUNCTION :

If $h(x)$ and $g(x)$ are differentiable functions of x then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f[h(x)].\,h'(x) - f[g(x)].\,g'(x)$$

5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\,[f(a) + f(a+h) + f(a+2h) + \ldots.. + f(a + \overline{n-1}h)]$$

$$= \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh) \text{ where } b-a = nh$$

If $a = 0$ and $b = 1$ then, $\lim_{n\to\infty} h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)\,dx$; where $nh = 1$

Or $\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx.$

6. ESTIMATION OF DEFINITE INTEGRAL :

(a) For a monotonic decreasing function in (a, b);

$f(b).(b-a) < \int_a^b f(x)\,dx < f(a).\,(b-a)$ and

(b) For a monotonic increasing function in (a, b);

$f(a).(b-a) < \int_a^b f(x)\,dx < f(b).(b-a)$

7. SOME IMPORTANT EXPANSIONS :

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ \infty = \ln 2$

(b) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ \infty = \frac{\pi^2}{6}$

(c) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ \infty = \frac{\pi^2}{12}$

(d) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ \infty = \frac{\pi^2}{8}$

(e) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots\ \infty = \frac{\pi^2}{24}$

EXERCISE - 1

Only One Correct Answer

1. The absolute value of $\dfrac{\int_0^{\pi/2} (x\cos x + 1)\,e^{\sin x}\,dx}{\int_0^{\pi/2} (x\sin x - 1)\,e^{\cos x}\,dx}$ is equal to :

(a) e (b) πe (c) $e/2$ (d) π/e

2. If x satisfies the equation $\left(\int_0^1 \frac{dt}{t^2+2t\cos\alpha+1}\right)x^2 - \left(\int_{-3}^{3} \frac{t^2 \sin 2t}{t^2+1}dt\right)x - 2 = 0$

$(0 < \alpha < \pi)$, then the value of x is :

(a) $\pm\sqrt{\frac{\alpha}{2\sin\alpha}}$ (b) $\pm\sqrt{\frac{2\sin\alpha}{\alpha}}$ (c) $\pm\sqrt{\frac{\alpha}{\sin\alpha}}$ (d) $\pm 2\sqrt{\frac{\sin\alpha}{\alpha}}$

3. Suppose the function $g_n(x) = x^{2n+1} + a_n x + b_n$ $(n \in N)$ satisfies the equation $\int_{-1}^{1}(px+q)\,g_n(x)\,dx = 0$ for all linear functions $(px+q)$ then :

(a) $a_n = b_n = 0$ (b) $b_n = 0; a_n = -\frac{3}{2n+3}$

(c) $a_n = 0; b_n = -\frac{3}{2n+3}$ (d) $a_n = \frac{3}{2n+3}; b_n = -\frac{3}{2n+3}$

4. Let $I(a) = \int_0^{\pi}\left(\frac{x}{a} + a\sin x\right)^2 dx$ where 'a' is positive real. The value of 'a' for which $I(a)$ attains its minimum value is :

(a) $\sqrt{\pi\sqrt{\frac{2}{3}}}$ (b) $\sqrt{\pi\sqrt{\frac{3}{2}}}$ (c) $\sqrt{\frac{\pi}{16}}$ (d) $\sqrt{\frac{\pi}{13}}$

5. Let $T = \int_0^{\ln 2} \frac{2e^{3x}+e^{2x}-1}{e^{3x}+e^{2x}-e^x+1}dx$, then e^T equals :

(a) $\frac{7}{4}$ (b) $\frac{7}{2}$ (c) $\frac{11}{2}$ (d) $\frac{11}{4}$

6. If the value of the definite integral $\int_0^1 {}^{207}C_7 x^{200}.(1-x)^7\,dx$ is equal to $\frac{1}{k}$ where $k \in N$. The value of 'k' is equal to :

(a) 208 (b) 210 (c) 212 (d) 214

7. The value of $\sqrt{\pi\left(\int_0^{2008} x|\sin \pi x|\,dx\right)}$ is equal to :

(a) $\sqrt{2008}$ (b) $\pi\sqrt{2008}$ (c) 1004 (d) 2008

8. The interval [0, 4] is divided into n equal sub-intervals by the points $x_0, x_1, x_2, ...,$ x_{n-1}, x_n where $0 = x_0 < x_1 < x_2 < x_3 ..... < x_n = 4$. If $\delta x = x_i - x_{i-1}$ for $i = 1, 2, 3, ... n$ then $\lim_{\delta x \to 0}\sum_{i=1}^{n} x_i \delta x$ is equal to :

(a) 4 (b) 8 (c) $\frac{32}{3}$ (d) 16

9. The absolute value of $\int_{10}^{19} \frac{(\sin x)}{(1+x^8)} dx$ is less than :

(a) 10^{-10} (b) 10^{-11} (c) 10^{-7} (d) 10^{-9}

10. The value of the definite integral $\int_0^{\pi/3} \ln(1+\sqrt{3}\tan x)\,dx$ equals :

(a) $\frac{\pi}{3}\ln 2$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi^2}{6}\ln 2$ (d) $\frac{\pi}{2}\ln 2$

11. Consider a function of the form $f(x) = \alpha e^{2x} + \beta e^{x} - \gamma x$, where α, β, γ are independent of x and $f(x)$ satisfies the following conditions $f(0) = -1$, $f'(\ln 2) = 30$ and $\int_0^{\ln 4} (f(x) + \gamma x)\,dx = 24$. The value of $(\alpha + \beta + \gamma)$ is equal to :

(a) 3 (b) 4 (c) 6 (d) 8

12. Let a function $h(x)$ be defined as $h(x) = 0$, for all $x \neq 0$. Also $\int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx = f(0)$, for every function $f(x)$. Then the value of the definite integral $\int_{-\infty}^{\infty} h'(x) \cdot \sin x\,dx$, is :

(a) equal to zero (b) equal to 1 (c) equal to -1 (d) non-existent

13. $\lim_{\lambda \to 0} \left(\int_0^1 (1+x)^{\lambda}\,dx \right)^{1/\lambda}$ is equal to :

(a) $2\ln 2$ (b) $\frac{4}{e}$ (c) $\ln\frac{4}{e}$ (d) 4

14. If $\beta + 2\int_0^1 x^2 e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx$ then the value of β is :

(a) e^{-1} (b) e
(c) $1/2e$ (d) cannot be determined

15. The true solution set of the inequality, $\sqrt{5x-6-x^2} + \left(\frac{\pi}{2}\int_0^x dz\right) > x\int_0^{\pi} \sin^2 x\,dx$ is :

(a) R (b) $(1, 6)$ (c) $(-6, 1)$ (d) $(2, 3)$

16. Let a, b and c be positive constants. The value of 'a' in terms of 'c' if the value of integral $\int_0^1 (acx^{b+1} + a^3 bx^{3b+5})\,dx$ is independent of b, equals :

(a) $\sqrt{\frac{3c}{2}}$ (b) $\sqrt{\frac{2c}{3}}$ (c) $\sqrt{\frac{c}{3}}$ (d) $\sqrt{\frac{3}{2c}}$

17. The value of the definite integral $\int_{e}^{e^{2010}} \frac{1}{x}\left(1+\frac{1-\ln x}{\ln x \ln\left(\frac{x}{\ln x}\right)}\right) dx$, equals :

(a) $2009 - \ln(2010 - \ln 2010)$
(b) $2010 - \ln(2009 - \ln 2009)$
(c) $2009 - \ln(2010 - \ln 2009)$
(d) $2010 - \ln(2010 - \ln 2010)$

18. For $U_n = \int_0^1 x^n (2-x)^n \, dx$; $V_n = \int_0^1 x^n (1-x)^n \, dx$ $n \in N$, which of the following statement(s) is/are true?

(a) $U_n = 2^n V_n$
(b) $U_n = 2^{-n} V_n$
(c) $U_n = 2^{2n} V_n$
(d) $U_n = 2^{-2n} V_n$

19. Let $I_1 = \int_0^{\pi/2} e^{-x^2} \sin(x)\, dx$; $I_2 = \int_0^{\pi/2} e^{-x^2} dx$; $I_3 = \int_0^{\pi/2} e^{-x^2}(1+x)\, dx$

and consider the statements

I $I_1 < I_2$

II $I_2 < I_3$

III $I_1 = I_3$

which of the following is(are) true?

(a) I only
(b) II only
(c) Neither I nor II nor III
(d) Both I and II

20. Let $S_n = \frac{n}{(n+1)(n+2)} + \frac{n}{(n+2)(n+4)} + \frac{n}{(n+3)(n+6)} + \dots + \frac{1}{6n}$, then $\lim_{n\to\infty} S_n$ has the value equal to :

(a) $\ln\frac{3}{2}$
(b) $\ln\frac{9}{2}$
(c) $2\ln\frac{3}{2}$
(d) $\frac{1}{2}\ln\frac{3}{2}$

21. The value of the definite integral, $\int_0^{\pi} \frac{\text{in } 5x}{\text{in } x} dx$ is :

(a) 0
(b) $\frac{\pi}{2}$
(c) π
(d) 2π

22. Consider the polynomial $f(x) = ax^2 + b$ c. If $f(0) = 0, f(2) = 2$ then the minimum value of $\int_0^2 |f'(x)|\, dx$ equals :

(a) 0
(b) 1
(c) 2
(d) none of these

23. If $A = [a_{ij}]_{n \times n}$, where $a_{ij} = i^{100} + j^{100}$, n $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_{ii}}{n^{101}}$ equals :

(a) $\frac{1}{50}$
(b) $\frac{1}{101}$
(c) $\frac{2}{101}$
(d) $\frac{3}{101}$

24. If $\sum_{i=1}^{6}(\sin^{-1}x_i+\cos^{-1}y_i)=9\pi$, then $\int\limits_{\sum_{i=1}^{6}x_i}^{\sum_{i=1}^{6}y_i} x\ln(1+x^2)\left(\frac{e^x}{1+e^{2x}}\right)dx$ is equal to :

(a) 0 (b) e^6+e^{-6} (c) $\ln\left(\frac{37}{12}\right)$ (d) e^6-e^{-6}

25. The value of definite integral $\int_{-a}^{a}\frac{x^2\cos x+e^x}{e^x+1}dx$ is equal to :

(a) $2a\cos(a)-a+(a^2-2)\sin(a)$ (b) $2a\cos(a)+a+(a^2-2)\sin(a)$
(c) $2a\cos(a)-a-(a^2-2)\sin(a)$ (d) $2a\cos(a)+a+(a^2+2)\sin(a)$

26. If $L=\lim_{n\to\infty}\frac{1}{n^4}\prod_{i=1}^{2n}(n^2+i^2)^{\frac{1}{n}}$, then $\ln L$ is equal to :

(a) $\ln 2+\frac{\pi}{2}-2$ (b) $2\tan^{-1}2-4+2\ln 5$
(c) $2\tan^{-1}2+4+2\ln 5$ (d) $2\ln 5-4-2\tan^{-1}2$

27. Let $f(x)=\begin{cases}\int_{-1}^{x}|t-2|dt; & x\neq 2\\ k; & x=2\end{cases}$. If $f(x)$ is continuous at $x=2$, then the value of k is equal to :

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{9}{2}$ (d) $\frac{7}{2}$

28. The value of the definite integral $\int_{-1}^{1}e^{-x^4}(1+\ln(x+\sqrt{x^2+1})+5x^3-4x^4)\,dx$ is equal to :

(a) $4e$ (b) $\frac{4}{e}$ (c) $2e$ (d) $\frac{2}{e}$

29. $\int_{0}^{1}\frac{(3x^4+4x^3+3x^2)}{(4x^3+3x^2+2x+1)^2}dx$ is equal to :

(a) $\frac{1}{100}$ (b) $\frac{1}{10}$ (c) $\frac{1}{20}$ (d) 0

30. A function $f(x)$ defined for $x\geq 0$ satisfies $\lim_{x\to\infty}\frac{f(x)}{x}=1$ and $f(0)=2$. Then $\int_{0}^{\infty}(f(x)-f'(x))e^{-x}\,dx$ equals :

(a) e^{-2} (b) 0 (c) 1 (d) 2

31. Consider a parabola $y = \frac{x^2}{4}$ and the point $F(0, 1)$.

Let $A_1(x_1, y_1), A_2(x_2, y_2), A_3(x_3, y_3), \ldots\ldots, A_n(x_n, y_n)$ are 'n' points on the parabola such as $x_k > 0$ and $\angle OFA_k = \frac{k\pi}{2n}$ $(k = 1, 2, 3, \ldots, n)$. Then the value of $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} FA_k$, is equal to :

(a) $\frac{2}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{8}{\pi}$ (d) none of these

32. For $n \in N$, the value of $\lim_{n\to\infty} \int_0^{\pi} x\,|\sin 2nx|\,dx$ equals :

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) $\frac{\pi}{4}$

33. Let f be a differentiable bijective function satisfying $\int_1^{f(x)} f^{-1}(t)\,dt = \frac{1}{3}\left(x^{\frac{3}{2}} - 8\right) \forall x > 0$ and $f(1) = 0$, then the value of $f(9)$ is :

(a) 3 (b) 9 (c) 0 (d) 2

34. The value of definite integral $\int_{-\pi}^{\pi} \frac{x^2}{1 + \sin x + \sqrt{1 + \sin^2 x}}\,dx$, is :

(a) $\frac{3\pi^3}{2}$ (b) $\frac{\pi^3}{3}$ (c) $\frac{2\pi^3}{3}$ (d) $\frac{\pi^3}{6}$

35. The value of definite integral $\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x^2 - x + 1}\,dx$ is equal to :

(a) $\frac{\pi^2}{6\sqrt{3}}$ (b) $\frac{\pi^2}{3\sqrt{3}}$ (c) $\frac{\pi^2}{12\sqrt{3}}$ (d) $\frac{\pi^2}{4\sqrt{3}}$

36. $\lim_{x\to 0} \frac{\int_0^x (t^2 + e^{t^2})^{\frac{1}{1-\cos t}}}{(e^x - 1)}\,dt$ is equal to :

(a) e^4 (b) e^2 (c) e^3 (d) e

37. $\lim_{n\to\infty} \left[(1+n)^{\frac{1}{n}} \left(1 + \frac{n}{2}\right)^{\frac{2}{n}} \left(1 + \frac{n}{3}\right)^{\frac{3}{n}} \ldots\ldots 2\right]^{\frac{1}{n}}$ is equal to :

(a) e (b) $e^{\frac{1}{2}}$ (c) $e^{\frac{1}{4}}$ (d) $e^{\frac{-1}{2}}$

38. The value of the definite integral $\int_{-1}^{1} \frac{dx}{1 + x^3 + \sqrt{1 + x^6}}$ equals :

(a) 2 (b) 1 (c) $\frac{1}{2}$ (d) $\sqrt{2}$

39. The value of definite integral $\frac{10}{\pi}\int_{\frac{-3\pi}{2}}^{\frac{-\pi}{2}}[(x+\pi)^3+\cos^2(x+3\pi)]\,dx$ is equal to :

(a) 4 (b) 5 (c) 6 (d) 7

40. The maximum value of $f(x)=\int_0^1 t\sin(x+\pi t)\,dt$, is :

(a) $\frac{1}{\pi}\sqrt{\pi^2+4}$ (b) $\frac{1}{\pi^2}\sqrt{\pi^2+4}$ (c) $\sqrt{\pi^2+4}$ (d) $\frac{1}{2\pi^2}\sqrt{\pi^2+4}$

41. If $\int_0^{2\pi}\frac{dx}{2+\sin 2x}=k\int_0^{\frac{\pi}{2}}\frac{dt}{7+\cos 2t}$, then the value of k, is :

(a) 4 (b) 8 (c) 12 (d) 16

42. The value of the definite integral $\int_0^1((e-1)\sqrt{\ln(1+(e-1)x)}+e^{x^2})\,dx$ is equal to :

(a) 0 (b) 1 (c) e (d) e^2

43. Let $a\in\left(0,\frac{\pi}{2}\right)$, then the value of $\lim_{a\to 0}\frac{1}{a^3}\int_0^a\ln(1+\tan a\tan x)\,dx$ is equal to :

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) 1

44. If $I=\int_0^1\frac{dx}{1+x^{\frac{\pi}{2}}}$, then :

(a) $\log_e 2<I<\frac{\pi}{4}$ (b) $\log_e 2>I$ (c) $I=\frac{\pi}{4}$ (d) $I=\log_e 2$

45. Let $f(x)=\begin{cases}\frac{1}{x^3}\int_0^x\sin(t^2)\,dt, & \text{for } x\neq 0\\ a, & \text{for } x=0\end{cases}$ $(a\in R)$

If $f(x)$ is continuous at $x=0$, then the value of a, is :

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{6}$

46. The value of the definite integral $\int_0^{\pi/2}(\cos^{10}x\cdot\sin 12x)\,dx$, is equal to :

(a) $\frac{1}{10}$ (b) $\frac{1}{11}$ (c) $\frac{1}{12}$ (d) $\frac{1}{22}$

47. Let $f(x)=\lim_{n\to\infty}((\cos x)^n+(\sin x)^n)^{\frac{1}{n}}$ for $x\in\left(0,\frac{\pi}{2}\right)$, then the value of $\int_0^{\pi/2}f(x)\,dx$ is equal to :

(a) $\sqrt{2}$ (b) $\sqrt{2}-1$ (c) 1 (d) $\frac{1}{\sqrt{2}}$

48. If $\int_{\pi/6}^{\pi/3} \frac{3\tan^2 x + 6\tan x + 11}{1+\tan^2 x} dx = \frac{k\pi + \lambda}{6}$, then the value of $(k + \lambda)$, is equal to :

(a) 10 (b) 12 (c) 14 (d) 16

49. Let $f: R \to [4, \infty)$ be an onto quadratic function whose leading coefficient is 1, such that $f'(x) + f'(2-x) = 0$. Then the value of $\int_1^3 \frac{dx}{f(x)}$, is equal to :

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

50. For real numbers a, b with $0 \le a \le \pi$, $a < b$. Let $I(a, b) = \int_a^b e^{-x} \sin x \, dx$.

If $\lim_{b\to\infty} I(a, b) = 0$, and the value of a is $k\pi$ $(k > 0)$, then the value of k, is :

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) 1

51. If $f(x) = pe^{2x} + qe^x + rx$ satisfies the condition $f(0) = -1$, $f'(\ln 2) = 31$ and $\int_0^{\ln 4} (f(x) - rx)\, dx = \frac{39}{2}$, then the value of $(p + q + r)$ is equal to :

(a) 0 (b) 2 (c) 3 (d) 4

52. Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x . \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$ then $\int_{-\pi}^{\pi} f(x)\, dx$ has the value equal to :

(a) $\frac{-\pi}{4}$ (b) $\frac{-\pi}{2}$ (c) $-\pi$ (d) -2π

53. The value of $\int_{\pi}^{2\pi} [2 \sin x]\, dx$ where [ ] represents the greatest integer function is :

(a) $-\frac{5\pi}{3}$ (b) $-\pi$ (c) $\frac{5\pi}{3}$ (d) -2π

54. The value of $\int_{\pi/2}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$ is equal to :

(a) $\frac{\pi}{2}$ (b) 2π (c) π (d) $\frac{\pi}{4}$

55. Let $V_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$. If $\sum_{r=1}^{100} V_r = \frac{k\pi}{2}$, where $k \in N$, then k is equal to :

(a) 100 (b) 2525 (c) 5050 (d) 4950

56. Let S be the area bounded by $y = e^{|\cos 4x|}$, $x = 0, y = 0$ and $x = \pi$.

Consider the following four relations

I : $S = 2\int_0^{\pi/2} e^{\cos t} dt$ II : $S = 2\int_0^{\pi/2} e^{\sin t} dt$ III : $S < 2(e^{\pi/2} - 1)$ IV : $S > \frac{\pi}{2}$

Number of relations which are correct?

(a) 1 (b) 2 (c) 3 (d) 4

57. Let $f(x)$ be a continuous and periodic function such that $f(x) = f(x + T)$ for all $x \in R, T > 0$. If $\int_{-2T}^{a+5T} f(x)\,dx = 19\ (a > 0)$ and $\int_0^T f(x)\,dx = 2$, then $\int_0^a f(x)\,dx$ is equal to :

(a) 3 (b) 5 (c) 7 (d) 9

58. $\int_0^{\pi/2} \frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)} dx$ is equal to :

(a) $\frac{\pi}{2} - \tan^{-1} 2$ (b) $\ln\frac{\pi}{2} - \ln(\tan^{-1} 2)$

(c) $\ln(\tan^{-1} 2)$ (d) $\ln\frac{\pi}{2}$

59. Let $\alpha > -1$ and $\beta > -1$, then the value of $\lim_{n\to\infty} n^{\beta-\alpha}\left(\frac{1^\alpha + 2^\alpha + \dots + n^\alpha}{1^\beta + 2^\beta + \dots + n^\beta}\right)$ is :

(a) $\frac{\beta+1}{\alpha+1}$ (b) $\frac{\alpha+1}{\beta+1}$

(c) $\frac{\alpha+2}{\beta+2}$ (d) $\frac{\beta+2}{\alpha+2}$

60. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \infty = \frac{\pi^2}{6}$ and $\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{3\pi^2}{k}$ then k equals :

(a) 72 (b) 36 (c) 24 (d) 12

61. A sequence of non-negative terms is given by $a_{n+1} = \int_{\frac{2}{3}(a_{n-1})}^{a_n} 3\,dx$ where $a_0 = 0, a_1 = 1$.

Then the value $(a_1 + a_2 + a_3 + \dots + a_{100})$ equals :

(a) 2^{100} (b) $2^{100} - 100$ (c) $2^{100} - 101$ (d) $2^{101} - 102$

62. The value of $\lim_{n\to\infty} \frac{(1^1 + 2^2 + \dots + n^2)(1^3 + 2^3 + \dots + n^3)(1^4 + 2^4 + \dots + n^4)}{(1^5 + 2^5 + \dots + n^5)^2}$ is equal to :

(a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

63. If x and y are independent variables and $f(x) = \left(\int_0^x e^{x-y} f'(y)\,dy\right) - (x^2 - x + 1)\,e^x$, where $f(x)$ is a differentiable function, then $f\left(\frac{-1}{2}\right)$ equals :

(a) $2\sqrt{e}$ (b) $-2\sqrt{e}$ (c) $\frac{-2}{\sqrt{e}}$ (d) $\frac{2}{\sqrt{e}}$

64. Let $f(x) = \int_1^x \frac{\ln t}{1+t}\,dt$; for $x > 0$, then the value of $f(e) + f\left(\frac{1}{e}\right)$ is :

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) none of these

65. Let $I_1 = \int_0^x e^{tx} \cdot e^{-t^2}\,dt$ and $I_2 = \int_0^x e^{-t^2/4}\,dt$ where $x > 0$ then the value of $\frac{I_1}{I_2}$ is :

(a) $e^{-x^2/2}$ (b) $e^{x^2/4}$ (c) $e^{-x^2/4}$ (d) $e^{x^2/2}$

66. Let $f(a) = \int_0^a \ln(1 + \tan a \tan x)\,dx$, then $f'\left(\frac{\pi}{4}\right)$ equals :

(a) $\frac{\pi}{4} + \frac{\ln 2}{2}$ (b) $\frac{\pi}{2} + \frac{\ln 2}{2}$ (c) $\frac{\pi}{4} + \ln 2$ (d) $\frac{\pi}{2} + \ln 2$

67. The value of the definite integral $\int_{-1}^{1} x \ln(1^x + 2^x + 3^x + 6^x)\,dx$ equals :

(a) $\frac{\ln 2 + \ln 3}{2}$ (b) $\frac{\ln 2 + \ln 3}{3}$ (c) $\ln 2 + \ln 3$ (d) $\frac{\ln 2 + \ln 3}{4}$

68. If $P = \lim_{n\to\infty} \frac{\left(\prod_{r=1}^{n}(n^3 + r^3)\right)^{1/n}}{n^3}$ and $\lambda = \int_0^1 \frac{dx}{1+x^3}$ then $\ln P$ is equal to :

(a) $\ln 2 - 1 + \lambda$ (b) $\ln 2 - 3 + 3\lambda$ (c) $2\ln 2 - \lambda$ (d) $\ln 4 - 3 + 3\lambda$

69. Consider $I_1 = \int_0^1 e^{-x} \cos^2 x\,dx$, $I_2 = \int_0^1 e^{-x^2} \cos^2 x\,dx$, $I_3 = \int_0^1 e^{\frac{-x^2}{2}} \cos^2 x\,dx$,

$I_4 = \int_0^1 e^{\frac{-x^2}{2}}\,dx$, then correct sequence is :

(a) $I_2 > I_4 > I_1 > I_3$ (b) $I_2 < I_4 < I_1 < I_3$

(c) $I_1 < I_2 < I_3 < I_4$ (d) $I_1 > I_2 > I_3 > I_4$

70. Let $f : [0, 1] \to R$ be a continuous function then the maximum value of $\int_0^1 f(x) \cdot x^2\,dx - \int_0^1 x \cdot (f(x))^2\,dx$ for all such function(s) is :

(a) $\frac{1}{8}$ (b) $\frac{1}{20}$ (c) $\frac{1}{12}$ (d) $\frac{1}{16}$

71. Let P_k be a point in xy plane whose x coordinate is $1+\frac{k}{n}$ $(k = 1, 2, 3, \ldots\ldots, n)$ on the curve $y = \ln x$. If A is (1, 0), then $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(AP_k)^2$ equals :

(a) $\frac{1}{3}+2\ln^2 2$ (b) $\frac{1}{3}+2\ln^2\left(\frac{2}{e}\right)$ (c) $\frac{1}{3}+\ln^2\left(\frac{2}{e}\right)$ (d) $\frac{1}{3}+2\ln\left(\frac{2}{e}\right)$

72. Given a function g continuous on R such that $\int_0^1 g(t)\,dt = 2$ and $g(1) = 5$.

If $f(x) = \frac{1}{2}\int_0^x (x-t)^2 g(t)\,dt$, then the value of $(f'''(1) - f''(1))$ is equal to :

(a) 0 (b) 3 (c) 5 (d) 7

73. Let $f: R \to R$ be defined as $f(x) = \int_0^x (\sin^2 t + 1)\,dt$ and g is the inverse function of f, then $g'\left(\frac{3\pi}{4}\right)$ is equal to :

(a) $\frac{4}{3\pi}$ (b) $\frac{4\pi}{3}$ (c) $\frac{1}{2}$ (d) 1

74. If $f(x) = x^3 + 3x + 4$ then the value of $\int_{-1}^{1} f(x)\,dx + \int_0^4 f^{-1}(x)\,dx$ equals :

(a) 4 (b) $\frac{17}{4}$ (c) $\frac{21}{4}$ (d) $\frac{23}{4}$

75. If $f(t) = \int_{2t}^{t^2} \tan^{-1}\left|\frac{(1+t)^2 - x}{1+x}\right| dx$ then minimum value of $f(x)$ is :

(a) $\frac{-\pi}{4}$ (b) $\frac{-\pi}{2}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

76. If $\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = A$ and $\int_0^\infty \left(\frac{x-\sin x}{x^3}\right) dx = \frac{aA}{b}$, where a and b are relative prime then the value of $(a + b)$ equals :

(a) 3 (b) 4 (c) 5 (d) 6

77. $\int_0^1 (\sqrt[4]{1-x^7} - \sqrt[7]{1-x^4})\,dx$ is equal to :

(a) $\frac{1}{2}$ (b) 1 (c) 0 (d) none of these

78. If $\int_x^{xy} f(t)dt$ is independent of x and $f(2) = 2$, then $\int_1^x \{f(t)\}$ is equal to :

(a) $2\ln x$ (b) $3\ln x$ (c) $4\ln x$ (d) none of these

79. Let $I_1 = \int_0^x e^{tx} \cdot e^{-t^2} dt$ and $I_2 = \int_0^x e^{-t^2/4} dt$ where $x > 0$ then the value of $\frac{I_1}{I_2}$ is :

(a) $e^{-x^2/2}$ (b) $e^{x^2/4}$ (c) $e^{-x^2/4}$ (d) $e^{x^2/2}$

80. A function $f(x)$ satisfies $f(x) = f\left(\frac{c}{x}\right)$ for some real number c (c > 1) and $\forall\, x > 0$. If $\int_1^{\sqrt{c}} \frac{f(x)}{x} dx = 3$, then the value of $\int_1^{c} \frac{f(x)}{x} dx$ is :

(a) 0 (b) 3 (c) – 3 (d) 6

81. The value of $\lim_{n\to\infty}\left\{\frac{1}{n^4}\left(\sum_{k=1}^{n} k^2 \int_k^{k+1} x\ln[(x-k)(k+1-x)]\,dx\right)\right\}$ is :

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{-1}{2}$ (d) – 1

82. The value of

$\int_{-\pi}^{\pi}(1+\cos x+\cos 2x+\ldots+\cos(2013x))\,(1+\sin x+\sin 2x+\ldots.+\sin(2013x))\,dx$, is :

(a) 0 (b) π (c) 2π (d) $2013\,\pi$

83. For real number u, $\frac{-\pi}{2} < \tan^{-1}u < \frac{\pi}{2}$ and $0 < \cot^{-1}u < \pi$, then the value of

$\frac{\int_0^1 \cot^{-1}(1-x+x^2)\,dx}{\int_0^1 \tan^{-1}x\;dx}$, is :

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) 2 (d) 1

84. Let g be a differentiable function satisfying $\int_0^x (x-t+1)\,g(t)\,dt = x^4 + x^2$ for all $x \geq 0$.

The value of $\int_0^1 \frac{12}{g'(x)+g(x)+10}\,dx$ is equal to :

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

85. The graph of $f(x) = x^2 + ax + b$ intersects the x-axis at 2 distinct points A, B and y-axis at C. The centroid of ΔABC lie on the line $y = x$. If $I = \int_0^6 f(x)\,dx$, then the value of I cannot be :

(a) – 50 (b) 50 (c) 100 (d) – 100

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 to 3

Suppose $f(x)$ and $g(x)$ are two continuous functions defined for $0 \le x \le 2$.

Given $f(x) = \int_0^1 e^{x+t} \cdot f(t)\, dt$ and $g(x) = \int_0^1 e^{x+t} \cdot g(t)\, dt + x$.

1. The value of $f(1)$ equals :
   (a) 0 (b) 1 (c) e^{-1} (d) e

2. The value of $g(0) - f(0)$ equals :
   (a) $\frac{2}{3-e^2}$ (b) $\frac{3}{e^2-2}$ (c) $\frac{1}{e^2-1}$ (d) 0

3. The value of $\frac{g(0)}{g(2)}$ equals :
   (a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{e^2}$ (d) $\frac{2}{e^2}$

Paragraph for Question Nos. 4 to 6

Consider $f(x) = 4x^4 - 24x^3 + 31x^2 + 6x - 8$ be a polynomial function and $\alpha, \beta, \gamma, \delta$ are the roots of the equation $f(x) = 0$, where $\alpha < \beta < \gamma < \delta$. Let sum of two roots of the equation $f(x) = 0$ vanishes.

4. The value of the expression $\delta^{\beta} + \frac{1}{\delta^{\alpha}} + \delta^{\gamma} + \gamma^{\delta}$ is :
   (a) 36 (b) 35 (c) 20 (d) 16

5. $\int \left(\frac{x-\delta}{x-\gamma}\right)^{\alpha+\beta+\delta} dx$ is :
   (a) $x - 8\ln|x-2| - \frac{24}{x-2} + \frac{16}{(x-2)^2} - \frac{16}{3(x-2)^3} + C$
   (b) $x - 16\ln|x-2| - \frac{24}{(x-2)} + \frac{32}{(x-2)^2} - \frac{16}{(x-2)^3} + C$
   (c) $x + 16\ln|x-2| - \frac{24}{x-2} - \frac{16}{(x-2)^2} + \frac{16}{3(x-2)^3} + C$
   (d) $x + 8\ln|x-2| - \frac{24}{x-2} - \frac{16}{(x-2)^2} + \frac{16}{3(x-2)^3} + C$

6. $\int_{2\alpha}^{2\beta} \frac{x^{\delta+1} - 5x^{\gamma+1} + 2\beta|x| + 1}{x^2 + 4\beta|x| + 1} dx$ is :

(a) $\ln 2$ (b) $2 \ln 2$ (c) $\frac{1}{2} \ln 2$ (d) $\ln \frac{1}{2}$

Paragraph for Question Nos. 7 to 9

Let f and g be two real-valued differentiable functions on R satisfying $\int_0^x g(t)\,dt = 3x + \int_x^0 \cos^2 t\, g(t)\,dt$ and $f(x) = \lim_{\alpha \to 0} \frac{1}{\alpha^4} \int_0^\alpha \frac{(e^{x+t} - e^x)\ln^2(1+t)}{2t^3 + 3} dt$.

7. $f(\ln 2)$ is greater than :

(a) 0 (b) $\frac{1}{6}$ (c) $\frac{3}{20}$ (d) $\frac{4}{25}$

8. Range of $g(x)$ is equal to :

(a) $\left[\frac{3}{2}, 3\right]$ (b) $\left[\frac{1}{2}, 3\right]$ (c) $\left[\frac{3}{2}, \infty\right)$ (d) $[-3, 3]$

9. The value of definite integral $\int_0^{\frac{\pi}{2}} g(x)\, dx$ lies in the interval :

(a) $\left(\frac{2\pi}{3}, \frac{4\pi}{5}\right)$ (b) $\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$ (c) $\left(\frac{3\pi}{4}, \frac{6\pi}{5}\right)$ (d) $\left(\pi, \frac{3\pi}{2}\right)$

Paragraph for Question Nos. 10 to 12

Let $g : R \to R$ be a differentiable function which satisfies $g(x) = 1 + \int_0^x g(t)\,dt$ and $g'(0) = 1$.

10. The value of $g(\ln 10) + g'(\ln 10) + g''(\ln 10)$ is equal to :

(a) 0 (b) $\frac{1}{10}$ (c) 30 (d) $\frac{1}{30}$

11. The value of definite integral $\int_{-3}^{-1} \left(\sum_{r=1}^{\infty} g(rx) \right) dx$ is equal to :

(a) $\ln(1 + e + e^{-1})$ (b) $\ln(1 + e^{-1} + e^{-2})$

(c) $\ln(1 + e + e^2)$ (d) $(1 + e^{-1} + e^2)$

12. Number of solution of the equation $g(-x) = g(x)$ is equal to :

(a) 0 (b) 1 (c) 2 (d) 3

Paragraph for Question Nos. 13 to 15

Let $P(x)$ be a polynomial function satisfying $P(x) - P'(x) = x^2 + 2x + 1$. Also

$$F(x) = \begin{cases} \left(\dfrac{P(x)}{10}\right)^{\frac{1}{\sin[3\sin(4x-4)]}}, & x \neq 1 \\ \dfrac{1}{e^{k^2-9k+40}}, & x = 1 \end{cases}$$

13. If $F(x)$ is continuous at $x = 1$, then the value of k can be :

(a) -4 (b) -5 (c) 4 (d) 5

14. The value of definite integral $\int_0^1 \frac{1}{P(x)}\,dx$ is also equal to :

(a) $\cot^{-1} 7$ (b) $\frac{3\pi}{4} - 2\tan^{-1} 2$ (c) $\frac{\pi}{4} - \tan^{-1} 2$ (d) $2\tan^{-1} 3 - \frac{3\pi}{4}$

15. Which of the following statement(s) is/are correct?

(a) Minimum value of $P(x)$ in $[0, 1]$ is 5

(b) The equation $P(x) = 0$ has non-real roots

(c) $|P(x)|$ is non-derivable at $x = 0$

(d) $P(|x|) > 0 \; \forall \; x \in R$

Paragraph for Question Nos. 16 to 18

Let f and g be two real-valued functions defined on R.

Given, $f(x) = \begin{cases} \dfrac{-2}{\pi^2}, & -\infty < x \leq 0 \\ \dfrac{-2}{\pi^2}\cos\pi & 0 < x < 1 \\ \dfrac{2}{\pi^2}, & 1 \leq x < \infty \end{cases}$ **and** $g(x) = 7 + 2x \ln 25 - 5^{x-1} - 5^{2-x}$.

Suppose α be the value of x for which the function $g(x)$ has the greatest value and β be the value of $\lim_{x \to 0} \dfrac{x^2 \sin(x - 2\pi)}{\int_0^x t^2\,dt}$.

16. Which one of the following statements is incorrect?

(a) Number of integers in the range of $f(x)$ is one

(b) $f(x)$ is continuous as well as differentiable for all real x

(c) $f(x)$ is non-monotonic on R

(d) The value of definite integral $\int_{\pi^2}^{5\pi^2} f(x)\,dx = 10$

17. If the function $h(x) = f(x)$ in the interval [0, 1] and $k(x) = h^{-1}(x)$, then the value of $k'(0)$ is equal to :

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{-\pi}{4}$ (d) $\frac{\pi}{6}$

18. The value of $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is equal to :

(a) 4 (b) 5 (c) 6 (d) 8

Paragraph for Question Nos. 19 and 20

Let a function 'f' satisfies $f(-x) = f(x)$ and $f(3+x) = f(1-x)\ \forall\ x \in R$ and

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1-2x, & 1 < x \le 2 \end{cases}.$$

19. The number of points where $f(x)$ is discontinuous in [0, 100], is :

(a) 100 (b) 50 (c) 25 (d) 0

20. The value of $\int_0^{100} f(x)\,dx$ is equal to :

(a) – 75 (b) – 50 (c) – 25 (d) 0

Paragraph for Question Nos. 21 and 22

Let a polynomial 'f' satisfies the relation $f(f(f(x))) + (1-p)f(x) = 3\ \forall\ x \in R$ where $p \in R$.

21. If leading coefficient of $f(x)$ is 2 then the value of $\frac{d}{dx}(f(f(x)))$ at $x = p$ is :

(a) 2 (b) 4 (c) 9 (d) 17

22. If leading coefficient of $f(x)$ is negative and $f(0) = 4$ then $\int_{-1}^{1} f^{-1}(x)\,dx$ is equal to :

(a) 8 (b) 16 (c) 32 (d) 64

Paragraph for Question Nos. 23 and 24

Let f be a monic polynomial satisfying $f(f(x)) = (f(x))^2 - 4\ \forall\ x \in R$ and

$$l = \lim_{t \to \sqrt{2}}\left(\frac{\int_2^{t^2}\frac{f(x)}{\ln(5-x^2)}\,dx}{\sin(t-\sqrt{2})}\right)$$ **where 'l' is a non-zero finite quantity.**

23. The value of l^2 is :

(a) 4 (b) $4\sqrt{2}$ (c) 8 (d) 16

24. The value of $\int_0^4 f(f(x))\, d(x^2)$ is :

(a) $\frac{-8}{3}$ (b) $\frac{-32}{3}$ (c) $\frac{16}{3}$ (d) $\frac{32}{3}$

Paragraph for Question Nos. 25 to 27

Let $y = g(x)$ be a function of the graph of broken line connected by points $(-2, 0)$, $(0, 3)$ and $(2, 4)$ in the x-y plane. Also $f(x) = \int_{-2}^{x} |g(t) - t|\, dt$, $x \in [-2, 2]$ and $h(x)$ is the inverse of $f(x)$.

25. The value of $f'(0)$ is :

(a) 3 (b) -3 (c) $\frac{1}{3}$ (d) does not exist

26. The value of $h\left(\frac{31}{4}\right) + h'\left(\frac{31}{4}\right)$ is :

(a) $\frac{17}{5}$ (b) $\frac{7}{5}$ (c) $\frac{9}{5}$ (d) $\frac{13}{5}$

27. The sum of roots of $f(x) = 2x^2 + 5$ is :

(a) 3 (b) $\frac{1}{3}$ (c) $\frac{4}{3}$ (d) $\frac{3}{4}$

Paragraph for Question Nos. 28 to 30

Let $f(x)$ be a periodic function. Another periodic function $g(x)$ can be obtained by compressing $f(x)$ by $\frac{1}{k_1}$ times in each of the period along x-axis and then magnifying it by k_2 times along y-axis.

28. If $f(x) = e^{\{x\}^2}(1 + 2\{x\}^2)$ and k_1, k_2 satisfy the equation $k_1^2 - 4k_1 + k_2^2 - 6k_2 + 13 = 0$, then the value of $\int_0^{500} g(x)\, dx$ is equal to :

(a) $750(e-1)$ (b) $1500(e-1)$ (c) $1500e$ (d) $750e$

29. If $f(x) = \text{sgn}(\cot^{-1} x) + \sin\frac{x}{3}$ and k_1, k_2 are integers satisfying the inequality $t^2 - 7t + 10 < 0$, $(k_1 < k_2)$ then $\int_0^{100\pi} g(x)\, dx$ is equal to :

(a) 200π (b) 400π (c) 600π (d) 800π

30. If $f(x) = 2 + \text{sgn}(x)(1 - \text{sgn}^2(x))$ and $h(x) = x^2 - 4x + 9$, $h(k_1) = k_2$ where $h'(k_1) = 0$ then the value of $\int_5^{15} g(x)\,dx$ is equal to :

(a) 25 (b) 50 (c) 100 (d) 200

[**Note :** $\{k\}$ and sgn (y) denote fractional part function of k and signum function of y respectively.]

Paragraph for Question Nos. 31 and 32

Let $f_n(x) = \sum_{n=1}^{n} \frac{\sin^2 x}{\cos^2\left(\frac{x}{2}\right) - \cos^2\left(\frac{2n+1}{2}\right)x}$ **and let** $g_n(x) = \prod_{n=1}^{n} f_n(x)$

$\forall\, n = 1, 2, 3, \ldots\ldots\ldots\ldots$

31. Let $I_n = \int_0^{\pi} \frac{f_n(x)}{g_n(x)}\,dx$. If $\sum_{k=1}^{100} I_n = k\pi$, then the value of k is :

(a) 50 (b) 25 (c) 100 (d) 75

32. The value of $\left[\lim_{x\to 0} \int_0^x \frac{9\,dt}{x f_9(t)\, g_9(t)}\right]$ is :

(a) 50 (b) 10 (c) 100 (d) 25

Paragraph for Question Nos. 33 and 34

Let $f: R \to R$ **be a bijective function. Let** $g(x)$ **be a continuous function such that** $\int_0^{f(x)} g(t)\,dt = g(f(x)) - 1\ \forall\ x \in R$

33. Value of $\int_0^1 g(\{x\}) \cdot \left[\frac{1-x}{(1+x)^3}\right] dx$ is :

[**Note :** $\{y\}$ denotes fractional part of y.]

(a) $1 - \frac{e}{4}$ (b) 1 (c) $1 + \frac{e}{4}$ (d) $\frac{e}{4} - 1$

34. The value of $\lim_{x\to\infty} e^{-x} \int_{\frac{1}{x}}^{x+\frac{1}{x}} g(t)\,dt$ equal to :

(a) 0 (b) 1 (c) 2 (d) DNE

Paragraph for Question Nos. 35 to 37

Consider $I_1 = \int_0^{\pi/2} \frac{x^3 \cos x\, dx}{(3\sin x - \sin 3x)}$, $I_2 = \int_0^{\pi/2} x^2 \operatorname{cosec}^2 x\, dx$ and $I_3 = \int_0^{\pi/2} x \cot x\, dx$

35. If $I_1 = \frac{\pi^3}{k_1} + k_2\, \pi \ln 2$, where $k_1, k_2 \in Q$ then $(k_1 + k_2)$ equals :

(a) $\frac{-509}{8}$ (b) $\frac{509}{8}$ (c) $\frac{511}{8}$ (d) $\frac{-511}{8}$

36. If $I_2 = k_1 + k_2\, \pi \ln 2$, where $k_1, k_2 \in Q$, then $(k_1^2 + k_2^2)$ equals :

(a) $\frac{1}{4}$ (b) $\frac{5}{32}$ (c) 1 (d) 9

37. If $I_3 = k_1 + k_2\, \pi \ln 2$, where $k_1 k_2 \in Q$, then $(k_1^3 + k_2^3)$ equals :

(a) $\frac{1}{8}$ (b) $\frac{1}{27}$ (c) $\frac{1}{64}$ (d) $\frac{3}{32}$

[**Note :** Q denotes the set of rational numbers.]

EXERCISE - 3

More Than One Correct Answers

1. Let $f(x)$ is a real valued function defined by : $f(x) = x^2 + x^2 \int_{-1}^{1} t \cdot f(t)\, dt + x^3 \int_{-1}^{1} f(t)\, dt$

then which of the following hold(s) good ?

(a) $\int_{-1}^{1} t \cdot f(t)\, dt = \frac{10}{11}$ (b) $f(1) + f(-1) = \frac{30}{11}$

(c) $\int_{-1}^{1} t \cdot f(t)\, dt > \int_{-1}^{1} f(t)\, dt$ (d) $f(1) - f(-1) = \frac{20}{11}$

2. Let $I_n = \int_0^{\pi/4} (\tan x)^n dx$ and let $J_n = (-1)^n I_{2n}$ for $n = 0, 1, 2$ then which of the following hold(s) good?

(a) $I_n + I_{n+2} = \frac{1}{n+1}$ (b) $J_n - J_{n-1} = \frac{(-1)^n}{2n-1}$ for $n \geq 1$

(c) If $u = \tan x$ then $I_n = \int_0^1 \frac{u^n}{1+u^2} du$ (d) $\lim_{n \to \infty} J_n = 0$

3. Let $S_n = \sum_{k=1}^{n} \frac{k^2+n^2}{n^3}$ and $T_n = \sum_{k=0}^{n-1} \frac{k^2+n^2}{n^3}$ for $n = 1,2,3,......$ then :

(a) $S_n < \frac{4}{3}$ (b) $T_n > \frac{4}{3}$ (c) $S_n > \frac{4}{3}$ (d) $T_n < \frac{4}{3}$

4. The value of the definite integral $\int_{-\infty}^{a} \frac{(\sin^{-1} e^x + \sec^{-1} e^{-x})\, dx}{(\tan^{-1} e^a + \tan^{-1} e^x)(e^x + e^{-x})}$ $(a \in R)$ is :

(a) independent of a (b) dependent on a

(c) $\frac{\pi}{2}\ln 2$ (d) $\frac{\pi}{2}\ln(2\tan^{-1} e^a)$

5. Which of the following definite integral(s) has/have their value equal to atleast one of the remaining three?

(a) $\int_0^{\infty} \frac{x}{1+x^4}\, dx$ (b) $\int_0^{\pi/4} \frac{x}{\cos x(\cos x + \sin x)}\, dx$

(c) $\int_1^{\frac{1+\sqrt{5}}{2}} \frac{x^2+1}{x^4-x^2+1} \ln\left(1+x-\frac{1}{x}\right) dx$ (d) $\int_0^1 \frac{\sin^{-1} x}{x}\, dx$

6. Let $f(x) = \begin{cases} x+1, & 0 \le x \le 1 \\ 2x^2 - 6x + 6, & 1 < x \le 2 \end{cases}$ and $g(t) = \int_{t-1}^{t} f(x)\, dx$ for $t \in [1, 2]$. Which of the following hold(s) good?

(a) $f(x)$ is continuous and differentiable in $[0, 2]$

(b) $g'(t)$ vanishes for $t = 3/2$ and 2

(c) $g(t)$ is maximum at $t = 3/2$

(d) $g(t)$ is minimum at $t = 1$

7. Let $f(x) = \int_0^x e^{t-[t]}\, dt\ (x > 0)$, where $[x]$ denotes greatest integer less than or equal to x, is :

(a) continuous and differentiable $\forall x \in (0, 3]$

(b) continuous but not differentiable $\forall x \in (0, 3]$

(c) $f(1) = e$ (d) $f(2) = 2(e-1)$

8. Let $l_1 = \lim_{x\to 0} \frac{1}{x^3} \int_0^x \frac{t(e^{3t}-1)\ln(1+2t)}{(t^3+3)(1-\cos\sqrt{t})}\, dt$ and

$l_2 = \lim_{x\to 0} \frac{1}{x}\left(\int_{\frac{\pi}{4}}^{v} \log_{\frac{1}{2}} \sin^2 t\, dt - \int_{\left(\frac{\pi}{4}+x\right)}^{v} \log_{\frac{1}{2}} \sin^2 t\, dt\right)$, where $\pi < v < 2\pi$. Then which of the following is(are) correct?

(a) $9l_1^2 + l_2^2 = 18$ (b) $3l_1 + 4l_2 = 8$ (c) $l_1 > 0$ and $l_2 < 0$ (d) $l_1 > l_2$

9. Let f be a real-valued function defined on R (the set of all real numbers) as $f(x) = \pi \left\{\frac{x}{\pi}\right\}$, then which of the following is(are) correct?

[**Note :** $\{k\}$ denotes the fractional part of k.]

(a) Range of $f(x)$ is $[0, \pi)$.

(b) $\lim\limits_{x \to \frac{\pi}{2}} f(x) = \frac{\pi}{2}$

(c) $\int_0^{2\pi} f(x)\,dx = \pi^2$

(d) $f'\left(\frac{5\pi}{2}\right) = 1$

10. Let $I_n = \lim\limits_{x \to \infty} \int_{e^{-x}}^{1} \left(\ln \frac{1}{t}\right)^n dt$ $(n = 1, 2, 3, \ldots)$ then which of the following is(are) incorrect?

(a) $I_n = 0$ if n is odd

(b) $I_n = 1$ if n is even

(c) $I_n = n!$ for all $n \in N$

(d) $I_n = e$ for all n

11. Let $I = \int_0^1 \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}\,dx$ and $J = \int_0^1 \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$, then the correct statement is :

(a) $I + J = 2$

(b) $I - J = \pi$

(c) $I = \frac{2+\pi}{2}$

(d) $J = \frac{4-\pi}{2}$

12. Let $f : R \to (0, \infty)$ be a real valued function satisfying $\int_0^x t\, f(x-t)\,dt = e^{2x} - 1$, then which of the following is(are) correct?

(a) The value $(f^{-1})'(4)$ equals $\frac{1}{8}$

(b) Derivative of $f(x)$ with respect to e^x at $x = 0$ is equal to 8

(c) The value of $\lim\limits_{x \to 0} \frac{f(x) - 4}{x}$ equals 4

(d) The value of $f(0)$ is equal to 4

13. Which of the following definite integral vanishes?

(a) $\int_{-\pi}^{\pi} (\cos 2x \cdot \cos 2^2 x \cdot \cos 2^3 x \cdot \cos 2^4 x \cdot \cos 2^5 x)\,dx$

(b) $\int_{-1}^{1} \ln(x + \sqrt{x^2 + 1})\,dx$

(c) $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$

(d) $\int_0^{\pi/2} \ln(\tan x)\,dx$

14. Let $f : [0, \infty) \to R$ be a continuous strictly increasing function, such that $f^3(x) = \int_0^x t \cdot f^2(t)\, dt$ for every $x \geq 0$. Which of the following is/are correct?

(a) $f(6)$ is equal to 6

(b) $f(x)$ is surjective

(c) $\int_0^1 f(x)\, dx$ is equal to $\frac{1}{18}$

(d) Number of solutions of equation $f(x) = 6$ are two.

15. Let $J = \int_0^{1/2} \left(\frac{1}{4} - x^2\right)^4 dx$ and $K = \int_0^{1/2} x^4 (1-x)^4\, dx$, then :

(a) $\frac{J}{K} = 2$

(b) $J - K = 0$

(c) $J = \frac{1}{2}\int_0^1 x^4 (1-x)^4\, dx$

(d) $K = \frac{1}{1260}$

16. If $f(\theta) = |||3\cos\theta - \sin\theta| + 2\cos\theta| + \cos\theta|$ where $\theta \in \left(\frac{\pi}{2}, \pi\right)$, then :

(a) $\int_{\pi/2}^{\pi} f(\theta)\, d\theta = 1$

(b) $f'\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$

(c) $\int_{\pi/2}^{\pi} f^2(\theta)\, d\theta = \frac{\pi}{4}$

(d) $f'\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$

17. Let f be a differentiable function on R and satisfying the integral equation $\int_0^x f(t)\, dt + \int_0^x t \cdot f(x-t)\, dt = -1 + e^{-x}$, for all $x \in R$, then :

(a) $f(2) = e^{-2}$ (b) $f(0) + f'(0) = 1$ (c) $f'(0) = 2$ (d) $f'(0) = 1$

18. Let $A = \int_1^{e^2} \frac{\ln x}{\sqrt{x}}\, dx$. Identify which of the following statement(s) is(are) correct?

(a) $A < 2\left(e - \frac{1}{e}\right)$

(b) $A < (e-1)\left(2 + \frac{1}{\sqrt{e}}\right)$

(c) $A = \int_0^2 t\sqrt{e^t}\, dt$

(d) $3 < A < 5$

19. Consider : $f(x) = 1 - 2p\cos x + p^2 = g(p)$; $h(x) = g(-p) \cdot g(p)$; $K(p) = \int_0^{\pi} \ln g(p)\, dx$ for $p, x \in R$. Then which of the following is(are) correct?

(a) $g(p^2) = h\left(\frac{x}{2}\right)$

(b) $K(p)$ is an even function

(c) $K(x) = \frac{1}{2}K(x^2)$

(d) $K(x) = \frac{1}{4}K(x^2)$

EXERCISE - 4

Match the Columns Type

1. Column I **Column II**

(a) The function $f(x) = \dfrac{e^{x\cos x} - 1 - x}{\sin x^2}$ is not defined at $x = 0$. (p) -1

The value of $f(0)$ so that f is continuous at $x = 0$ is

(b) The value of the definite integral $\int_0^1 \dfrac{dx}{\sqrt{x} + \sqrt[3]{x}}$ equals (q) 0

$a + b \ln 2$, where a and b are integers then $(a + b)$ equals

(c) Given $e^n \int_0^n \dfrac{\sec^2\theta - \tan\theta}{e^\theta} d\theta = 1$ then the value of $\tan(n)$ (r) $\dfrac{1}{2}$

is equal to

(d) Let $a_n = \int_{\frac{1}{n+1}}^{1/n} \tan^{-1}(nx)\, dx$ and $b_n = \int_{\frac{1}{n+1}}^{1/n} \sin^{-1}(nx)\, dx$ then (s) 1

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ has the value equal to

2. Column I **Column II**

(a) The value of the definite integral, $\int_{\theta_1}^{\theta_2} \dfrac{d\theta}{1 + \tan\theta} = \dfrac{501\pi}{K}$ (p) 0

where $\theta_2 = \dfrac{1003\pi}{2008}$ and $\theta_1 = \dfrac{\pi}{2008}$. The value of K equals

(b) Suppose that the function f, g, f' and g' are continuous over $[0, 1]$, $g(x) \neq 0$ for $x \in [0, 1]$, $f(0) = 0, g(0) = \pi$, (q) 2007

$f(1) = \dfrac{2009}{2}$ and $g(1) = 1$

The value of the definite integral,

$\int_0^1 \dfrac{f(x)\cdot g'(x)\{g^2(x) - 1\} + f'(x)\cdot g(x)\{g^2(x) + 1\}}{g^2(x)} dx$ is equal to

(c) Here is a problem that involves both 2007 and 2008 and is perfect for doing in New Year's eve. The value of the integral (r) 2008

$\int_0^1 \left(\sqrt[2007]{1 - x^{2008}} - \sqrt[2008]{1 - x^{2007}}\right) dx$, is equal to (s) 2009

3. Let $f_n(x) + f_n(y) = \dfrac{x^n + y^n}{x^n y^n}$ for all $x, y \in R - \{0\}$ where $n \in N$ and $g(x) = \max \{f_2(x), f_3(x), \frac{1}{2}\}, \forall x \in R - \{0\}$.

Column I	Column II
(a) The possible value(s) of $\sum_{k=1}^{\infty} f_{2k}(\operatorname{cosec} \theta) + \sum_{k=1}^{\infty} f_{2k}(\sec\theta)$ where $\theta \neq \frac{k\pi}{2}, k \in I$ is/are	(p) 1
(b) The value(s) of n for which $h(x) = f_n(x)$ sgn x is even, is/are	(q) 2
(c) Values of consecutive integers between which, $\int_{1/2}^{2} g(x)\, dx$ lie are	(r) 3
(d) Number of values of x for which the function $y = g(x)\ \forall x \in R - \{0\}$ is non-differentiable, is/are	(s) 4

4.

Column I	Column II
(a) Let $f(x)$ is a continuous function. If $f(1) = 1$ and $\int_0^x t \cdot f(2x - t)\, dt = \frac{1}{2}\tan^{-1}(x^2)$ then the value of $4\int_1^2 f(x)\, dx$ is equal to	(p) 0
(b) Let $f(x) = \prod_{r=1}^{2009}(x - r)$, then the value of the definite integral $\int_1^{2009} f(x)\, dx$ is equal to	(q) 1
(c) If $\int_0^{\ln 10} \frac{e^x\sqrt{e^x - 1}}{e^x + 8}\, dx$ has the value equal to $(a - b\pi)$ then $\frac{a}{b}$ equals	(r) 2
(d) Suppose g is the inverse function of a differentiable function f and $G(x) = \frac{1}{g(x)}$. If $f(4) = 2$ and $f'(4) = \frac{1}{16}$ then $(G'(2))^2$ equals	(s) 3
	(t) 4

5.

Column I	Column II
(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \cdot \sin\left(2\pi + \frac{3\pi i}{n}\right)$ is equal to	(p) $\frac{1}{e}$
(b) $\int_0^{\infty} [x]e^{-x} dx$ equals (where [ ] denotes the greatest integer function)	(q) $\frac{2}{e}$

(c) If $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{\ln 2i-\ln n}{n}\right)=\ln k$, then '$k$' equals (r) $\frac{1}{\pi}$

(d) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2}\sin\left(\frac{i^2\pi}{n^2}\right)$ is equal to (s) $\frac{2}{\pi}$

(t) $\frac{1}{e-1}$

6. Column I **Column II**

(a) Let $f(t)=\sqrt{1-\sin t}$, then $\int_0^{2\pi} f(t)\,dt-\int_0^{\pi} f(t)\,dt$, is equal to (p) 2

(b) For $x\neq 2$, if $\int_{4-x}^{x} e^{x(4-x)}dx=2$, then $\int_{4-x}^{x} xe^{x(4-x)}dx$ is equal to (q) 4

(c) Let f be a differentiable function on R satisfying (r) 6

$f(x)=x^2+\int_1^x t\,f(t)\,dt$. The value of $f'(1)$ is equal to (s) 3

7. Column I **Column II**

(a) If $I=\int_2^3((x-1)^3+(4-x)^3+x)\cos\pi x\,dx$, then $|50\pi^2 I|$ is equal to (p) 0

(b) If $J=\int_0^{10}\text{sgn}(\sin\pi x)\,dx$, then $10J$ is equal to, (q) 100

where sgn x denotes signum function of x

(c) If $K=\int_0^{102}[\cot^{-1}x]\,dx$, then $[K]$ is equal to, where $[y]$ (r) 50

denotes largest integer less than or equal to y

(d) If $L=\frac{\int_0^{51}[x+25]\,dx}{\int_0^{51}\{x+25\}\,dx}$, then $\frac{L}{2}$ is equal to, (s) 70

where $[y]$ and $\{y\}$ denote greatest integer function and fractional part function respectively.

8. Column I **Column II**

(a) Let $f(x-3)=f(x+3)$ for all $x\in R$ and (p) 0

$f(x)=\begin{cases}x, & 0\le x<3\\ 6-x, & 3\le x<6\end{cases}$. If $\int_0^{54} f(x)\,dx=9^k$ then k is equal to

(b) If $\lim_{x\to\frac{\pi}{2}} \dfrac{\left(\int_0^x \sin^2\theta\, d\theta\right)\left(\int_0^x \cos^2\theta\, d\theta\right) - \frac{\pi^2}{16}}{x - \frac{\pi}{2}} = \dfrac{\pi}{\lambda}$, then λ is equal to (q) 1

(c) If $L = \int_0^{\frac{9\pi}{2}} (1-\sin x)\cdot \sin(x+\cos x)\, dx$, then $[L]$ is equal to (r) 2

[**Note :** $[y]$ denotes greatest integer less than or equal to y.]

(d) If $M = \int_0^1 \sin^{-1}(1-x)\, dx + \int_2^3 \cos^{-1}(x-2)\, dx$, then [M] is equal to (s) 4

[**Note :** $[y]$ denotes greatest integer less than or equal to y.]

9. **Column I** **Column II**

(a) Given that $f(0)=0, f'(0)=1, f'(2)=3$ and $f''(2)=5$. (p) 2

The value of the definite integral $\int_0^1 x f'''(2x)\, dx$ is equal to

(b) The value of the definite integral (q) 6

$$6\int_0^1 \left(\frac{2^{\log 2^{1/4^x}} - 3^{\log 27^{(x^2+1)^3}} - 2x}{7^{4\log_{49} x} - x - 1}\right) dx \text{ equals}$$

(c) If the value of definite integral $\int_0^{2\pi} |x^2 - \pi^2 - \sin^2 x|\, dx = k\pi^3$, (r) 11

then k equals (s) none

EXERCISE - 5

Integer Answer Type

1. If a_1, a_2 and a_3 are the three values of a which satisfy the equation

$$\int_0^{\pi/2} (\sin x + a\cos x)^3\, dx - \frac{4a}{\pi-2}\int_0^{\pi/2} x\cos x\, dx = 2$$

then find the value of $1000(a_1^2 + a_2^2 + a_3^2)$.

2. If $\int_{\pi/4}^{\pi/3} \dfrac{(\sin^3\theta - \cos^3\theta - \cos^2\theta)(\sin\theta + \cos\theta + \cos^2\theta)^{2010}}{(\sin\theta)^{2012}(\cos\theta)^{2012}}\, d\theta = \dfrac{(a+\sqrt{b})^n - (1+\sqrt{c})^n}{d}$

where a, b, c and d are all positive integers. Find the value $(a+b+c+d)$.

3. For positive integers n, let $A_n = \frac{1}{n}\{(n+1)+(n+2)+ \ldots . + (n+n)\}$, $B_n = \{(n+1)(n+2) \ldots \ldots (n+n)\}^{1/n}$. If $\lim\limits_{n\to\infty} \frac{A_n}{B_n} = \frac{ae}{b}$ where $a, b \in N$ and relatively prime find the value of $(a+b)$.

4. Given that $U_n = \{x(1-x)\}^n$ and $n \geq 2$ and $\frac{d^2U_n}{dx^2} = n(n-1)U_{n-2} - 2n(2n-1)U_{n-1}$, further if $V_n = \int\limits_0^1 e^x \cdot U_n \, dx$, then for $n \geq 2$, we have $V_n + K_1 n(2n-1). V_{n-1} + K_2 n(n-1) V_{n-2} = 0$. Find $(K_1 + K_2)$.

5. If the value of the definite integral $\int\limits_{-1}^{1} \cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) \cdot \left(\cot^{-1} \frac{x}{\sqrt{1-(x^2)^{|x|}}}\right) dx = \frac{\pi^2(\sqrt{a}-\sqrt{b})}{\sqrt{c}}$, where $a, b, c \in N$ in their lowest form, then find the value of $(a+b+c)$.

6. Find the number of values of x satisfying $\int\limits_0^x t^2 \sin(x-t)\, dt = x^2$ in $[0, 100]$.

7. If the value of the integral $\int\limits_0^{\pi/2} (\cos x)^{2011} (\sin 2013x)\, dx$ is $\frac{a}{b}$ where a and b are co-prime then find the value of $(2a+b)$.

8. Let $f : (0, 1) \to (0, 1)$ be a differentiable function such that $f'(x) \neq 0$ for all $x \in (0, 1)$ and $f\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$. If $f(x) = \lim\limits_{t \to x} \frac{\int\limits_0^t \sqrt{1-f^2(s)}\, ds - \int\limits_0^x \sqrt{1-f^2(s)}\, ds}{f(t)-f(x)}$, then the value of $f\left(\frac{1}{4}\right)$ equals $\frac{\sqrt{m}}{4}$ where $m \in N$. Find the value of m.

9. If the value of definite integral $\frac{\int\limits_0^{\pi/2} (\cos x)^{\sqrt{2}+1} dx}{\int\limits_0^{\pi/2} (\cos x)^{\sqrt{2}-1} dx}$ is equal to $(n - \sqrt{n})$, where $n \in N$ then find the value of n.

10. Let $I = \int\limits_0^1 (1-x^{50})^{99} x^{100} dx$ and $J = \int\limits_0^1 (1-x^{50})^{100} x^{100} dx$. If $\frac{I}{J}$ is equal to $\frac{m}{n}$ where m and n are co-prime then find the value of $\left(\frac{m-n-1}{20}\right)$.

11. Find the value of definite integral $\int_1^2 \left(\frac{e^x - e^{2/x}}{x}\right) dx$.

12. Let n be a positive integer. Define $f(x) = \min(|x-1|, |x-2|, \ldots, |x-n|)$. If $\int_0^{n+1} f(x)\,dx = 2$, then find the value of n.

13. Let $f(x) = x\cos x, x \in \left[\frac{3\pi}{2}, 2\pi\right]$ and g is the inverse function of f. If $\int_0^{2\pi} g(x)\,dx = a\pi^2 + b\pi + c$, where $a, b, c \in R$, then find the value of $2(a+b+c)$.

14. Let $g : R \to \{4\}$ be a function given by $g(x) = x^3(f'(t)-2) + x^2 f''(t) + 4x(f(0)+6) + 4$ and $h(x)$ is defined as $h(x) = \begin{cases} \int_0^x |f(t)-2|\,dt, & 0 \le x \le 6 \\ (x-6)^2 + 20, & 6 < x \le 12 \end{cases}$. If number of integers in the range of $h(x)$ is N then find the value of $\frac{N}{19}$.

15. Let $I_n = \int_{-1}^{1} |x| \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^{2n}}{2n}\right) dx$. If $\lim_{n\to\infty} I_n$ can be expressed as rational $\frac{p}{q}$ in the lowest form, then find the value of $pq\,(p^3 + q^2)$.

16. Let f be a function defined by $f(x) = \begin{cases} (x-r)^2, & r-1 \le x < r+1 \\ 1, & r+1 \le x \le r+2 \end{cases}$, where $r = 3k, k \in I$.

Find the value of $\int_0^{45} f(x)\,dx$.

17. If $\int_0^1 \frac{dx}{\sqrt{1+x} + \sqrt{1-x} + 2}$ can be expressed in the form $a\sqrt{b} - \frac{\pi}{c} - 1$ where a, b, c are prime numbers. Find the value of $(a+b+c)$.

18. If $\int_0^2 \frac{\ln(1+2x)}{1+x^2}$ $(\tan^{-1} a)(\ln\sqrt{b})$ where $a, b \in N$, find the value of $(a^2 + b^2)$.

19. Given a function g, continuous everywhere such that $g(1) = 5$ and $\int_0^1 g(t)\,dt = 2$. If $f(x) = \frac{1}{2}\int_0^x (x-t)^2\, g(t)\,dt$, then compute the value of $f'''(1) - f''(1)$.

20. Let $I_n = \int_0^{\pi/4} \tan^n x\,dx\ (n = 0, 1, 2, 3, 4, \ldots)$

and $S_n = \sum_{n=0}^{n} (I_n I_{n+1} + I_n I_{n+3} + I_{n+1} I_{n+2} + I_{n+2} I_{n+3})$. Find the value of $\lim_{n\to\infty} 100(S_n)$.

21. Given $\lim_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{1/n} = \frac{a}{b}$ where a and b are relatively prime, find the value of $(a+b)$.

22. If $f: R \to R$ is a continuous and differentiable function such that,
$\int_{-1}^{x} f(t)\,dt + f'''(3)\int_{x}^{0} dt = \int_{1}^{x} t^3 dt - f'(1)\int_{x}^{2} t^2 dt + f''(2)\int_{3}^{x} t\,dt$, then find the value of $f'(4)$.

23. Let a be a real number. If the value of definite integral $\int_{-\pi+a}^{3\pi+a} |x - a - \pi| \sin\left(\frac{x}{2}\right) dx$ is equal to -16 then sum of all the values of a in $[0, 314]$ is $k\pi$. Find the value of k.

24. If $\int_{0}^{\pi/2} \frac{\sin^2(10)\theta}{\sin^2\theta} d\theta = k\pi$, where $k \in N$ then find the value of k.

25. Let k be a positive integer and $f(x)$ be a polynomial with integer coefficients satisfying $2\int_{1}^{x} f(t)\,dt + x^k = x f(x)$, where $x \geq 1$. Find the sum of all possible values of k.

26. If $\int_{1/5}^{5} \frac{1}{x}\left(\left\{\frac{x}{4x^2 - 2x + 9}\right\} + \left\{-\frac{x}{9x^2 - 2x + 4}\right\}\right) dx = p \ln q + 3\tan^{-1} r$, where $\{\cdot\}$ denotes fractional part function, p and q are relatively prime numbers and r is an integer, then find the value of $p + q + r$.

27. Let $f(x)$ be a function satisfying $f(x) = f\left(\frac{100}{x}\right) \forall x > 0$. If $\int_{1}^{10} \frac{f(x)}{x} dx = 5$ then find the value of $\int_{1}^{100} \frac{f(x)}{x} dx$.

28. Let $I_n = \int_{0}^{\pi/2} x(\cos x + \sin x)^n dx$, find the value of $\frac{101\, I_{101} - \frac{\pi}{2}}{I_{99}}$.

29. Let $F(n) = \int \frac{x^n}{e^x} dx$, then find the value of definite integral $\int_{2}^{5}\left(e^{\int \frac{F(1)}{F(2)} dx}\right)^2 dx$, where $F(1)$, $F(2)$ and $\int \frac{F(1)}{F(2)} dx$ contains no constant term.

30. If the value of definite integral $\int_{0}^{2} \frac{ax + b}{(x^2 + 5x + 6)^2} dx$ is equal to $\frac{7}{30}$, then find the value of $(a^2 + b^2)$.

31. Let $f(x)$ and $g(x)$ are 2 differentiable functions satisfying
$f(x) + 3g(x) = x^2 + x + 6$

$2f(x) + 4g(x) = 2x^2 + 4$

If $J = \int_0^{\pi/4} \ln(g(\tan^2 x) - f(\tan x) - 8)\, dx$, then find the value of $\frac{3\pi \ln 2}{J}$.

32. If the function $\int_0^x f(t)\, dt \to 5$ as $|x| \to 1$, where f is continuous then find the number of integers in the range of p so that the equation $2x + \int_0^x f(t)\, dt = p$ has two roots of opposite sign in $(-1, 1)$.

33. If $\int_0^{\pi/4} \frac{\ln(\cot x)}{((\sin x)^{2009} + (\cos x)^{2009})^2} . (\sin 2x)^{2008}\, dx = \frac{a^b \ln a}{c^2}$ (where a, b, c are in their lowest form) then find the value of $(a + b + c)$.

34. If the value of the definite integral $I = \int_1^{\infty} \frac{(2x^3 - 1)\, dx}{x^6 + 2x^3 + 9x^2 + 1}$ can be expressed in the form $\frac{A}{B} \cot^{-1} \frac{C}{D}$ where $\frac{A}{B}$ and $\frac{C}{D}$ are rationals in their lowest form, find the value of $(A + B^2 + C^3 + D^4)$.

35. Let a and b be two positive real numbers. Find the value of the definite integral $\int_a^b \frac{e^{x/a} - e^{b/x}}{x}\, dx$.

36. Let $J = \int_0^1 \frac{2x - (1 + x^2)^2 \cot^{-1} x}{(1 + x^2)(1 - (1 + x^2)\cot^{-1} x)}\, dx$. Find the value of $100\,(J - \ln 2)$.

37. If $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k \ln\left(\frac{n^2 + (k-1)^2}{n^2 + k^2}\right)$ exists and is equal to L. Find the absolute value of $[L]$.

[Note : $[y]$ denotes greatest integer less than or equals to y.]

38. If $\lim_{t \to \infty} \left(\int_1^t \frac{dx}{1 + \sqrt[3]{x^2}} - a t^b\right)$ exists and equals to non-zero finite number L, where a and b are positive real numbers, then find the value of $(ab - 4L - 3\pi)$.

39. If $\int_0^{\pi} \frac{x}{\sqrt{1 + \sin^3 x}} ((3\pi \cos x + 4 \sin x) \sin^2 x + 4)\, dx = k\pi^2$, then find the value of k.

40. Let $f(k) = \int_{-1}^{1} \frac{\sqrt{1 - x^2}}{\sqrt{k + 1 - x}}\, dx$. If $\sum_{k=0}^{99} f(k) = p\pi$, then find the value of p.

41. Let $f(x) = e^x + 2x + 1$ then find the value of $\int_2^{e+3} f^{-1}(x)\, dx$.

42. Find the value of $\int_0^{\pi} \frac{x^2\cos^2 x - x\sin x - \cos x - 1}{(1+x\sin x)^2}\,dx$.

43. Let $f(x)$ be a continuous function defined from $[0, 2] \to R$ and satisfying the equation $\int_0^2 f(x)\,(x - f(x))\,dx = \frac{2}{3}$. Find the value of $2f(1)$.

44. If the value of definite integral $\int_{\pi/4}^{65\pi/4} \frac{dx}{(1+2^{\cos x})(1+2^{\sin x})}$ equals $k\pi$, where $k \in N$ then find k.

45. If $\frac{\int_0^1 (1-(1-x^2)^{100})^{201} . x\,dx}{\int_0^1 (1-(1-x^2)^{100})^{202} . x\,dx} = \frac{p}{q}$ where $p, q \in N$, then find the least value of $(p-q)$.

46. If x_1 and x_2 $(x_1 < x_2)$ are two values of x satisfying the equation

$$\left|2\left(x^2+\frac{1}{x^2}\right)+|1-x^2|\right| = 4\left(\frac{3}{2}-2^{x^2-3}-\frac{1}{2^{x^2+1}}\right),$$

then find the value of $\int_{x_1+x_2}^{3x_2-x_1} \left\{\frac{x}{4}\right\}\left(1+\left[\tan\left(\frac{\{x\}}{1+\{x\}}\right)\right]\right)dx$.

[**Note** : $[y]$ and $\{y\}$ denote greatest integer and fractional part functions respectively.]

47. Let f be a differentiable function defined such that $f : [0, 27] \to \left[\frac{1}{3}, 6\right]$ and $f'(x) < 0$ $\forall x \in D_f$. If $\int_0^{27} x f'(x)\,dx = \lambda - 3\int_0^3 x^2 f(x^3)\,dx$ then find the value of λ.

[**Note** : D_f denotes the domain of the function.]

48. Let $g(x)$ be a real valued function defined on the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ such that $g(x) = e^{2x} + \int_0^{\sin x} \frac{e^t}{\cos^2 x + 2t\sin x - t^2}\,dt \;\forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Also $f(x)$ be the inverse function of $g(x)$, where $0 \le x \le \frac{\pi}{2}$. Find the value of $\frac{1}{(f'(1))^2} + g(0) + g'(0) + g''(0)$.

49. Let $P(x)$ be a polynomial with real coefficients such that $(x^2+x+1)\,P(x-1) = (x^2-x+1)\,P(x)\;\forall x \in R$ and $P(1) = 3$. If $\int_0^1 \tan^{-1}\left(\frac{2x}{1+P(x^2)}\right)dx = \int_0^1 \tan^{-1}(x+1)\,dx = \frac{k}{16}(\pi - \ln 4)$ then find the value of k.

50. Let $f(x) = 10^{10x}$, $g(x) = \log_{10}\left(\frac{x}{10}\right) \forall x \in (0, \infty)$. Also $h_1(x) = g(f(x))$ and $h_n(x) = h_1(h_{n-1}(x)), \forall n \geq 2$.

Let $I_n = \int_{-1}^{1}(h_n(x) - h_{(n+1)}(x))\,dx$, , $n = 1, 2, 3$. If the value of $\left(\frac{9}{20}\sum_{n=1}^{2011} I_n + 1\right)$ is 10^N, then find N, where N is a natural number.

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (a)	2. (d)	3. (b)	4. (a)	5. (d)	6. (a)	7. (d)	8. (b)	9. (c)	10. (a)
11. (a)	12. (c)	13. (b)	14. (a)	15. (d)	16. (a)	17. (a)	18. (c)	19. (d)	20. (a)
21. (b)	22. (c)	23. (c)	24. (a)	25. (b)	26. (b)	27. (c)	28. (d)	29. (b)	30. (d)
31. (b)	32. (b)	33. (d)	34. (b)	35. (a)	36. (a)	37. (b)	38. (b)	39. (b)	40. (b)
41. (d)	42. (c)	43. (b)	44. (a)	45. (c)	46. (b)	47. (a)	48. (d)	49. (d)	50. (c)
51. (b)	52. (c)	53. (b)	54. (c)	55. (c)	56. (d)	57. (b)	58. (b)	59. (a)	60. (b)
61. (d)	62. (b)	63. (c)	64. (b)	65. (b)	66. (a)	67. (b)	68. (b)	69. (c)	70. (d)
71. (b)	72. (b)	73. (c)	74. (d)	75. (a)	76. (c)	77. (c)	78. (c)	79. (b)	80. (d)
81. (c)	82. (c)	83. (c)	84. (c)	85. (b)					

EXERCISE 2 : Linked Comprehension Type

1. (a)	2. (a)	3. (b)	4. (a)	5. (a)	6. (b)	7. (acd)	8. (a)	9. (cd)	10. (c)
11. (b)	12. (b)	13. (cd)	14. (abd)	15. (abd)	16. (d)	17. (b)	18. (c)	19. (b)	20. (a)
21. (b)	22. (c)	23. (c)	24. (c)	25. (a)	26. (b)	27. (c)	28. (c)	29. (b)	30. (c)
31. (a)	32. (c)	33. (a)	34. (b)	35. (a)	36. (c)	37. (a)			

EXERCISE 3 : More Than One Correct Answers

1. (b, d)	2. (a, b, c, d)	3. (c, d)	4. (a, c)	5. (b, c)
6. (b, c, d)	7. (b, d)	8. (b, d)	9. (a, b, c, d)	10. (a, b, d)
11. (b, d)	12. (a, b, d)	13. (a, b, c, d)	14. (a, c)	15. (b, c, d)
16. (a, b, c, d)	17. (a, b, c)	18. (a, b, c, d)	19. (a, b, c)	

EXERCISE 4 : Match the Columns Type

1. (a) (r), (b) (p), (c) (s), (d) (r)
2. (a) (r), (b) (s), (c) (p)
3. (a) (r) (s), (b) (p) (r), (c) (q) (r), (d) (r)
4. (a) (s), (b) (p), (c) (t), (d) (q)
5. (a) (s), (b) (t), (c) (q), (d) (r)
6. (a) (q), (b) (q), (c) (s)
7. (a) (q), (b) (p), (c) (p), (d) (r)
8. (a) (r), (b) (s), (c) (p), (d) (q)
9. (a) (p), (b) (r), (c) (p)

EXERCISE 5 : Integer Answer Type

1. 5250	2. 2024	3. 11	4. 1	5. 7
6. 16	7. 2014	8. 7	9. 2	10. 5
11. 0	12. 0005	13. 3	14. 3	15. 186
16. 25	17. 6	18. 29	19. 3	20. 100
21. 43	22. 10	23. 1200	24. 5	25. 8
26. 7	27. 10	28. 200	29. 66	30. 116
31. 8	32. 2	33. 4019	34. 99	35. 0
36. 100	37. 1	38. 13	39. 2	40. 10
41. 2	42. 0	43. 1	44. 4	45. 1
46. 2	47. 9	48. 18	49. 12	50. 2011

□□□

Chapter 7

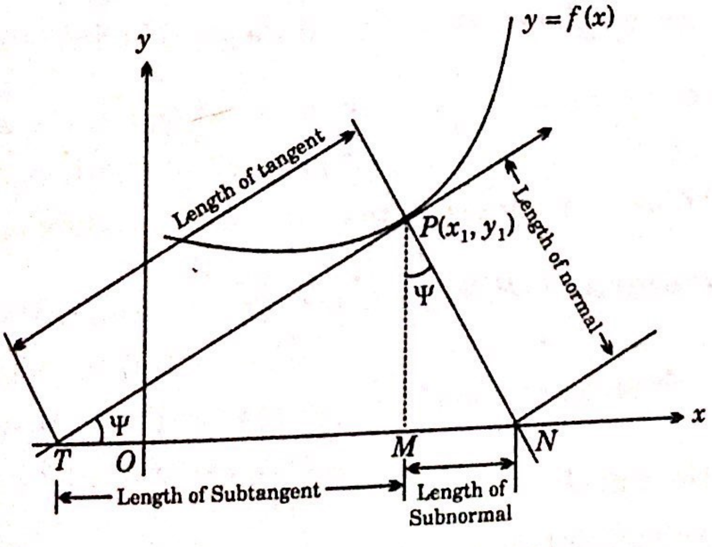
Application of Derivatives

KEY CONCEPTS

TANGENT AND NORMAL

I. The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically $f'(x_1) = \left.\frac{dy}{dx}\right]_{x_1y_1}$ = Slope of tangent at $P(x_1 y_1) = m$ (say).

II. Equation of tangent at (x_1, y_1) is; $y - y_1 = \left.\frac{dy}{dx}\right]_{x_1y_1} (x - x_1)$.



III. Equation of normal at (x_1, y_1) is; $y - y_1 = -\frac{1}{\left.\frac{dy}{dx}\right]_{x_1y_1}}(x - x_1)$.

Note :

1. The point $P(x_1, y_1)$ will satisfy the equation of the curve and the equation of tangent and normal line.
2. If the tangent at any point P on the curve is parallel to the axis of x then $\frac{dy}{dx} = 0$ at the point P.
3. If the tangent at any point on the curve is parallel to the axis of y, then $\frac{dy}{dx} = \infty$ or $\frac{dx}{dy} = 0$.
4. If the tangent at any point on the curve is equally inclined to both the axes then $\frac{dy}{dx} = \pm 1$.
5. If the tangent at any point makes equal intercept on the co-ordinate axes then $\frac{dy}{dx} = -1$.
6. Tangent to a curve at the point $P(x_1, y_1)$ can be drawn even through $\frac{dy}{dx}$ at P does not exist, *e.g.* $x = 0$ is a tangent to $y = x^{1/3}$ at (0, 0).
7. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation, *e.g.* If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ *i.e.* $x + y = 0$ and $x - y = 0$.

IV. Angle of intersection between two curves is defined as the angle between the 2 tangents drawn to the 2 curves at their point of intersection. If the angle between two curves is 90° every where then they are called **Orthogonal** curves.

V. (a) Length of the tangent $(PT) = \frac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent $(MT) = \frac{y_1}{f'(x_1)}$

(c) Length of Normal $(PN) = y_1\sqrt{1+[f'(x_1)]^2}$

(d) Length of Subnormal $(MN) = y_1 f'(x_1)$

VI. Differentials :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, $y = \tan x$ then $dy = \sec^2 x\, dx$.

In general $dy = f'(x)\, dx$.

Note :

$d(c) = 0$ where 'c' is a constant.

$d(u + v - w) = du + dv - dw$

$d(uv) = udv + vdu$

Note :

1. For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' *i.e.* $\Delta y \neq dy$.
2. The relation $dy = f'(x)\, dx$ can be written as $\frac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

MONOTONICITY

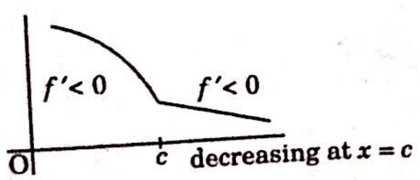
(Significance of the sign of the first order derivative)

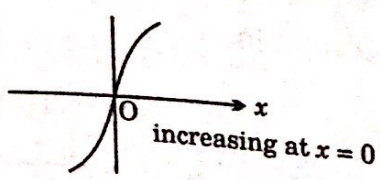
DEFINITIONS :

1. A function $f(x)$ is called an Increasing Function at a point $x = a$ if in a sufficiently small neighbourhood around $x = a$ we have $\left.\begin{matrix} f(a+h) > f(a) \text{ and} \\ f(a-h) < f(a) \end{matrix}\right\}$ increasing;

   Similarly decreasing if $\left.\begin{matrix} f(a+h) < f(a) \text{ and} \\ f(a-h) > f(a) \end{matrix}\right\}$ decreasing.

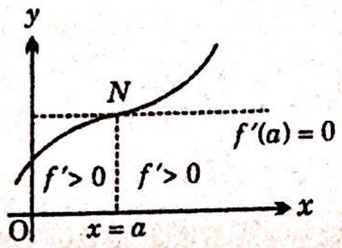
2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.
3. A function which in a given interval is increasing or decreasing is called **"Monotonic"** in that interval.
4. **Tests for increasing and decreasing of a function at a point :** If the derivative $f'(x)$ is positive at a point $x = a$, then the function $f(x)$ at this point is increasing. If it is negative, then the function is decreasing. Even if $f'(a)$ is not defined, f can still be increasing or decreasing.

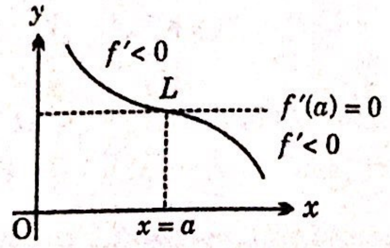




Note : If $f'(a) = 0$, then for $x = a$ the function may be still increasing or it may be decreasing as shown. It has to be identified by a separate rule. *e.g.* $f(x) = x^3$ is increasing at every point.

Note that, $\dfrac{dy}{dx} = 3x^2$.





5. Tests for Increasing and Decreasing of a function in an interval :

Sufficiency Test : If the derivative function $f'(x)$ in an interval (a, b) is every where positive, then the function $f(x)$ in this interval is Increasing;

If $f'(x)$ is every where negative, then $f(x)$ is Decreasing.

Note :

1. If a continuous function is invertible it has to be either increasing or decreasing.
2. If a function is continuous the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.
3. If f is increasing in $[a, b]$ and is continuous then f(b) is the greatest and f(c) is the least value of f in $[a, b]$. Similarly if f is decreasing in $[a, b]$ then $f(a)$ is the greatest value and $f(b)$ is the least value.

6. (a) ROLLE'S THEOREM :

Let $f(x)$ be a function of x subject to the following conditions :

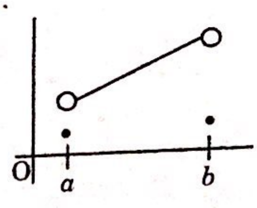
(i) $f(x)$ is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = 0$.

Note that if f is not continuous in closed $[a, b]$ then it may lead to the adjacent graph where all the 3 conditions of Rolle's will be valid but the assertion will not be true in (a, b).



(b) LMVT THEOREM :

Let $f(x)$ be a function of x subject to the following conditions :

(i) $f(x)$ is a continuous function of x in the closed interval of $a \leq x \leq b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) \neq f(b)$. Then there exists at least one point $x = c$ such that $a < c < b$

where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically, the slope of the secant line joining the curve at $x = a$ and $x = b$ is equal to the slope of the tangent line drawn to the curve at $x = c$. Note the following :

- Rolle's theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

Note : Now $[f(b) - f(a)]$ is the change in the function f as x changes from a to b so that $[f(b) - f(a)]/(b - a)$ is the *average rate of change* of the function over the interval $[a, b]$. Also $f'(c)$ is the actual rate of change of the function for $x = c$. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval. This interpretation of the theorem justifies the name "Mean Value" for the theorem.

(c) Application of Rolle's Theorem for Isolating the Real Roots of an Equation $f(x) = 0$

Suppose a and b are two real numbers such that;

(i) $f(x)$ and its first derivative $f'(x)$ are continuous for $a \leq x \leq b$.

(ii) $f(a)$ and $f(b)$ have opposite signs.

(iii) $f'(x)$ is different from zero for all values of x between a and b. Then there is one and only one real root of the equation $f(x) = 0$ between a and b.

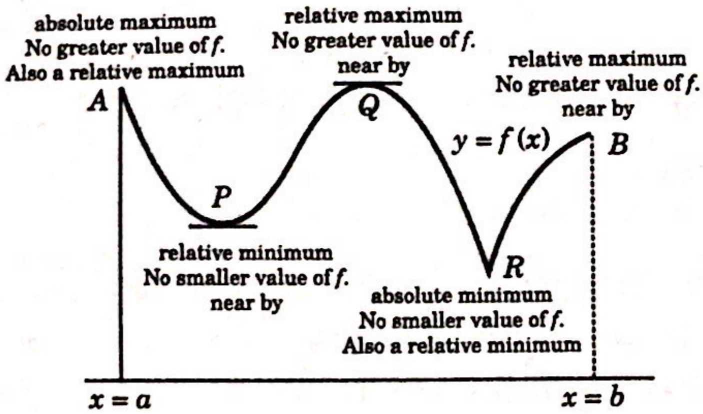
MAXIMA - MINIMA

FUNCTIONS OF A SINGLE VARIABLE

1. HOW MAXIMA AND MINIMA ARE CLASSIFIED

A function $f(x)$ is said to have a maximum at $x = a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = a$. Symbolically

$$\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right] \Rightarrow x = a \text{ gives maxima for a sufficiently small positive } h.$$



Similarly, a function $f(x)$ is said to have a minimum value at $x = b$ if $f(b)$ is least than every other value assumed by $f(x)$ in the immediate neighbourhood at $x = b$.

Symbolically if $\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right] \Rightarrow x = b$ gives minima for a sufficiently small positive h.

Note :

1. The maximum and minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest and least values of the function relative to some neighbourhood of the point in question.
2. The term 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.
3. A maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.
4. A function can have several maximum and minimum values and a minimum value may even be greater than a maximum value.

5. Maximum and minimum values of a continuous function occur alternately and between two consecutive maximum values there is a minimum value and *vice versa*.

2. A NECESSARY CONDITION FOR MAXIMUM AND MINIMUM :

If $f(x)$ is a maximum or minimum at $x = c$ and if $f'(c)$ exists then $f'(c) = 0$.

Note :

1. The set of values of x for which $f'(x) = 0$ are often called as stationary points or critical points. The rate of change of function is zero at a stationary point.
2. In case $f'(c)$ does not exist $f(c)$ may be a maximum or a minimum and in this case left hand and right hand derivatives are of opposite signs.
3. The greatest (global maxima) and the least (global minima) values of a function f in an interval $[a, b]$ are $f(a)$ or $f(b)$ or are given by the values of x for which $f'(x) = 0$.
4. Critical points are those points in the domain of function where either $\frac{dy}{dx} = 0$ or it fails to exist.

3. SUFFICIENT CONDITION FOR EXTREME VALUES :

$$\left.\begin{matrix} f'(c-h) > 0 \\ f'(c+h) < 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local maxima, where } f'(c) = 0.$$

Similarly $$\left.\begin{matrix} f'(c-h) < 0 \\ f'(c+h) > 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local minima, where } f'(c) = 0.$$

h is a sufficiently small positive quantity

Note : If $f'(x)$ does not change sign *i.e.* has the same sign in a certain complete neighbourhood of c, then $f(x)$ is either strictly increasing or decreasing throughout this neighbourhood implying that $f(c)$ is not an extreme value of f.

4. USE OF SECOND ORDER DERIVATIVE IN ASCERTAINING THE MAXIMA OR MINIMA :

(a) $f(c)$ is a minimum value of the function f, if $f'(c) = 0$ and $f''(c) > 0$.

(b) $f(c)$ is a maximum value of the function f, $f'(c) = 0$ and $f''(c) < 0$.

Note : If $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaining the maxima or minima.

5. SUMMARY-WORKING RULE :

FIRST : When possible, draw a figure to illustrate the problem and label those parts that are important in the problem. Constants and variables should be clearly distinguished.

SECOND : Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. Its may require some algebraic manipulations.

THIRD : If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $\frac{dy}{dx} = f'(x) = 0$.

FOURTH : Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

(a) If $\frac{d^2y}{dx^2}$ is positive when $\frac{dy}{dx} = 0 \Rightarrow y$ is minimum.

If $\frac{d^2y}{dx^2}$ is negative when $\frac{dy}{dx} = 0 \Rightarrow y$ is maximum.

If $\frac{d^2y}{dx^2} = 0$ when $\frac{dy}{dx} = 0$, the test fails.

(b) If $\frac{dy}{dx}$ $\left.\begin{array}{lll} \text{positive} & \text{for} & x < x_0 \\ \text{is zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{array}\right] \Rightarrow$ a maximum occurs at $x = x_0$.

But if $\frac{dy}{dx}$ changes sign from negative to zero to positive as x advances through x_0 there is a minimum. If $\frac{dy}{dx}$ does not change sign, neither a maximum nor a minimum. Such points are called **Inflection Points**.

FIFTH : If the function $y = f(x)$ is defined for only a limited range of values $a \le x \le b$ then examine $x = a$ and $x = b$ for possible extreme values.

SIXTH : If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Note :

1. Given a fixed point $A\ (x_1, y_1)$ and a moving point $P\ (x, f(x))$ on the curve $y = f(x)$. Then AP will be maximum or minimum if it is normal to the curve at P.

2. If the sum of two positive numbers x and y is constant then their product is maximum if they are equal, *i.e.* $x + y = c, x > 0, y > 0$, then $xy = \frac{1}{4}((x+y)^2 - (x-y)^2)$

3. If the product of two positive numbers is constant then their sum is least if they are equal, *i.e.* $(x+y)^2 = (x-y)^2 + 4xy$

6. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

(a) Volume of a cuboid = lbh.

(b) Surface area of a cuboid = $2(lb + bh + hl)$.

(c) Volume of a prism = area of the base x height.

(d) Lateral surface of a prism = perimeter of the base x height.

(e) Total surface of a prism = lateral surface + 2 area of the base
(Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.

(g) Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.

(Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(i) Curved surface of a cylinder = $2\pi rh$.

(j) Total surface of a cylinder = $2\pi rh + 2\pi r^2$.

(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(l) Surface area of a sphere = $4\pi r^2$.

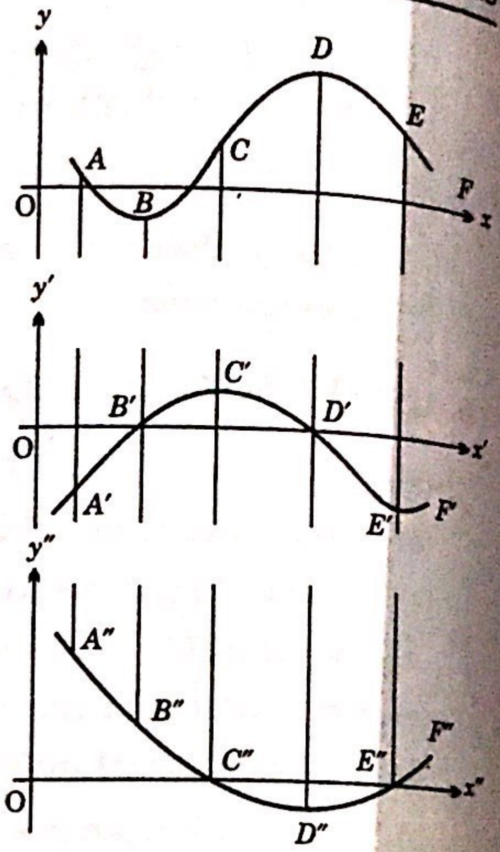
(m) Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

7. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINTS OF INFLECTION :

The sign of the 2nd order derivative determines the concavity of the curve. Such points such as C and E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if:

(a) $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave upwards

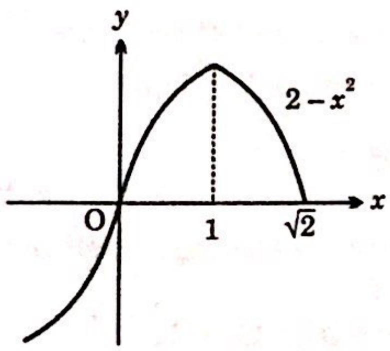
(b) $\dfrac{d^2y}{dx^2} < 0 \Rightarrow$ concave downwards.



At the point of inflection we find that $\dfrac{d^2y}{dx^2} = 0$ and $\dfrac{d^2y}{dx^2}$ changes sign.

Inflection points can also occur if $\dfrac{d^2y}{dx^2}$ fails to exist. For example, consider the graph of the function defined as,

$$f(x) = \begin{cases} x^{3/5} & \text{for } x \in (-\infty, 1) \\ 2 - x^2 & \text{for } x \in (1, \infty) \end{cases}$$



Note that the graph exhibits two critical points one is a point of local maximum and the other a point of inflection.

EXERCISE - 1

Only One Correct Answer

1. The values of p for which the function $f(x) = \left(\dfrac{\sqrt{p+4}}{1-p} - 1\right)x^5 - 3x + \ln 5$ decreases for all real x is :

(a) $(-\infty, \infty)$

(b) $\left[-4, \dfrac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$

(c) $\left[-3, \dfrac{5-\sqrt{27}}{2}\right] \cup (2, \infty)$

(d) $[1, \infty)$

2. If $P(x) = (2013)\, x^{2012} - (2012)\, x^{2011} - 16x + 8$, then $P(x) = 0$ for $x \in \left[0, 8^{\frac{1}{2011}}\right]$ has :

(a) exactly one real root
(b) no real root
(c) atleast one and at most two real roots
(d) atleast two real roots

3. The set of value(s) of 'a' for which the function $f(x) = \dfrac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ possess a negative point of inflection .

(a) $(-\infty, -2) \cup (0, \infty)$
(b) $\{-4/5\}$
(c) $(-2, 0)$
(d) empty set

4. The minimum value of $\left(1 + \dfrac{1}{\sin^n \alpha}\right)\left(1 + \dfrac{1}{\cos^n \alpha}\right)$ is :

(a) 1
(b) 2
(c) $(1 + 2^{n/2})^2$
(d) none of these

5. Let $f(x) = e^{|x^2 - 4x + 3|}$ then :

(a) $f(x)$ decreases in the interval $(1, 2) \cup (3, \infty)$
(b) $f(x)$ increases in the interval $(-\infty, 1) \cup (2, 3)$
(c) $f(x)$ has one local maximum point and two local minimum points
(d) $f(x)$ has one local minimum point and two local maximum points

6. A circle with centre at $(15, -3)$ is tangent to $y = \dfrac{x^2}{3}$ at a point in the first quadrant. The radius of the circle is equal to :

(a) $5\sqrt{6}$
(b) $8\sqrt{3}$
(c) $9\sqrt{2}$
(d) $6\sqrt{5}$

7. If f be a continuous function on $[0, 1]$, differentiable in $(0, 1)$ such that $f(1) = 0$, then their exists some $c \in (0, 1)$ such that :

(a) $c f'(c) - f(c) = 0$
(b) $f'(c) + c f(c) = 0$
(c) $f'(c) - c f(c) = 0$
(d) $c f'(c) + f(c) = 0$

8. If $y = f(x)$ is twice differentiable function such that $f(a) = f(b) = 0$, and $f(x) > 0\ \forall\ x \in (a, b)$, then :

(a) $f''(c) < 0$ for some $c \in (a, b)$
(b) $f''(c) > 0\ \forall\ c \in (a, b)$
(c) $f(c) = 0$ for some $c \in (a, b)$
(d) none of these

9. The value of t for which $\dfrac{\int_0^{\pi/2} (\sin x + t \cos x)\, dx}{\sqrt{\int_0^{\pi/2} (\sin x + t\cos x)^2\, dx}}$ is maximum lies in the interval :

(a) $\left(0, \dfrac{1}{2}\right)$
(b) $\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$
(c) $\left(\dfrac{3}{4}, \dfrac{3}{2}\right)$
(d) $\left(\dfrac{3}{2}, \dfrac{5}{2}\right)$

10. Let f, g and h are differentiable function such that $g(x) = f(x) - x$ and $h(x) = f(x) - x^3$ are both strictly increasing functions, then the function $F(x) = f(x) - \frac{\sqrt{3}x^2}{2}$ is :

(a) strictly increasing $\forall\, x \in R$

(b) strictly decreasing $\forall\, x \in R$

(c) strictly decreasing on $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and strictly increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$

(d) strictly increasing on $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and strictly decreasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$

11. The global maximum value of $f(x) = \cot x - \sqrt{2}\,\text{cosec}\, x$ in interval $(0, \pi)$ is equal to :

(a) 1 (b) -1 (c) 0 (d) non-existent

12. Let α be a fixed constant number such that $0 < \alpha < \frac{\pi}{2}$. The function F is defined by $F(\theta) = \int_0^\theta x \cos(x + \alpha)\, dx$. If θ lies in the range of $\left[0, \frac{\pi}{2}\right]$, then the maximum value of $F(\theta)$, is :

(a) $\frac{\pi}{2} - \alpha + \sin\alpha$ (b) $\frac{\pi}{2} - \alpha + \cos\alpha$ (c) $\frac{\pi}{2} - \alpha - \sin\alpha$ (d) $\frac{\pi}{2} - \alpha - \cos\alpha$

13. The complete set of non-zero values of k such that the equation $|x^2 - 10x + 9| = kx$ is satisfied by atleast one and atmost three values of x, is :

(a) $(-\infty, -16] \cup [4, \infty)$ (b) $(-\infty, -16] \cup [16, \infty)$

(c) $(-\infty, -4] \cup [4, \infty)$ (d) $(-\infty, 4] \cup [16, \infty)$

14. If $f(x) = \int_0^1 e^{|t-x|}\, dt$ where $(0 \le x \le 1)$, then maximum value of $f(x)$ is :

(a) $e - 2$ (b) $e - 3$ (c) $e - 1$ (d) $2(\sqrt{e} - 1)$

15. If $\alpha, \beta \in \left(\frac{\pi}{2}, \pi\right)$ and $\alpha < \beta$, then which one of the following is true?

(a) $e^{\cos\alpha - \cos\beta} < \frac{\alpha}{\beta}$ (b) $e^{\cos\beta - \cos\alpha} < \frac{\beta}{\alpha}$

(c) $e^{\cos\alpha - \cos\beta} < \frac{\beta}{\alpha}$ (d) $e^{\cos\beta - \cos\alpha} < \frac{\alpha}{\beta}$

16. The maximum value of $(\cos\alpha_1) \cdot (\cos\alpha_2) \ldots\ldots (\cos\alpha_n)$, under the restrictions $0 \le \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \le \frac{\pi}{2}$ and $\cot\alpha_1 \cdot \cot\alpha_2 \ldots\ldots \cot\alpha_n = 1$ is :

(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2n}$ (d) 1

17. If $a_1, a_2, ..., a_n$ are positive real numbers whose product is a fixed number e, the minimum value of $a_1 + a_2 + a_3 + ... + a_{n-1} + 2a_n$ is :

(a) $n(2e)^{1/n}$ (b) $(n+1)e^{1/n}$

(c) $2ne^{1/n}$ (d) $(n+1)(2e)^{1/n}$

18. Let $f(x) = 2x^3 - 3(2+p)x^2 + 12px + \ln(16 - p^2)$. If $f(x)$ has exactly one local maxima and one local minima, then the number of integral values of p is :

(a) 4 (b) 5 (c) 6 (d) 7

19. Let $f(x) = \left(\frac{a^2 - 4}{a^2 + 2}\right)x^3 - 3x + \sin 3x$, then the true set of values of a for which $f(x)$ is strictly decreasing on R, is :

(a) $(-\infty, -2)$ (b) $[2, \infty)$

(c) $[-2, 2]$ (d) $(5, \infty)$

20. If $p \in (p_1, p_2)$ satisfy the condition that the point of local minimum and point of local maximum is less than 4 and greater than -2 respectively for the function $f(x) = x^3 - 3px^2 + 3(p^2 - 1)x + 1$, then the value of $(p_2 - p_1)$ is equal to :

(a) -1 (b) 3 (c) 0 (d) 4

21. The smallest natural number c for which the equation $e^x = cx^2$ has exactly three real and distinct solutions, is

(a) 1 (b) 2 (c) 3 (d) 4

22. If the equation $x^4 + 8x^3 + 18x^2 + 8x + a = 0$ has four distinct real roots, then the range of a is :

(a) $(0, 9)$ (b) $(-9, 0)$ (c) $(-8, 1)$ (d) $(-1, 8)$

23. Let $f(x) = (pq - q^2 - 1)x + \int_0^x (\cos^4\theta + \sin^4\theta)\, d\theta$. If $f(x)$ is strictly decreasing $\forall\, q \in R$, $\forall\, x \in R$, then the least integral value of p, is :

(a) 0 (b) 1 (c) 2 (d) 3

24. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies the Rolle's theorem in $[1, 3]$ and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then the values of a and b are respectively.

(a) $-6, 1$ (b) $-2, 1$ (c) $1, -2$ (d) $1, -6$

25. A population $P(t)$ of 1000 bacteria introduced into nutrient medium grows according to law $P(t) = 1000 + \frac{1000t}{100 + t^2}$. The maximum size of bacterial population is equal to :

(a) 1100 (b) 1250 (c) 1050 (d) 5250

26. Let $f(x) = \left(1+\frac{1}{x}\right)^x$ and $g(x) = \left(1+\frac{1}{x}\right)^{x+1}$, both f and g being defined for $x > 0$, then prove that $f(x)$ is increasing and $g(x)$ is decreasing.

(a) Both $f(x)$ and $g(x)$ are increasing
(b) $f(x)$ is increasing and $g(x)$ is decreasing
(c) $f(x)$ is decreasing and $g(x)$ is increasing
(d) Both $f(x)$ and $g(x)$ are decreasing.

27. Let $f(x) = |x-1| + |x-2|$, $I = \int_0^3 f(x)\,dx$, $M =$ the minimum value of f, $N = f'(x)$ for $x < -4$ and $C =$ the value of $f''(4)$. Then the value of $\frac{M^2 - N^2 + IC}{2}$ is :

(a) $\frac{-3}{2}$ (b) $\frac{-5}{2}$ (c) $\frac{3}{2}$ (d) $\frac{5}{2}$

28. A piece of paper in the shape of a sector of a circle (see figure 1) is rolled up to form a right-circular cone (see figure 2). The value of angle θ is :

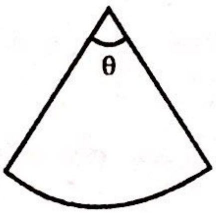


figure 1

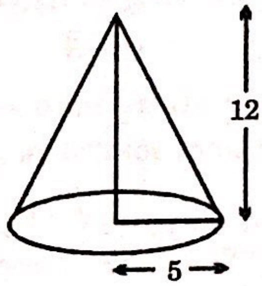


figure 2

(a) $\frac{10\pi}{13}$ (b) $\frac{9\pi}{13}$ (c) $\frac{5\pi}{13}$ (d) $\frac{6\pi}{13}$

29. Let θ be the acute angle between the curves $y = x^x \ln x$ and $y = \frac{2^x - 2}{\ln 2}$ at their point of intersection on the line $y = 0$. The value of $\tan\theta$ is equal to :

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{1}{3}$ (d) 3

30. The range of real constant 't' such that $(1-t)\sin\theta + t\tan\theta > \theta$ always holds $\forall\, \theta \in \left(0, \frac{\pi}{2}\right)$ is :

(a) $[1, \infty)$ (b) $\left[\frac{1}{2}, \infty\right)$ (c) $\frac{1}{3}$ (d) none of these

31. If equation of tangent drawn to the curve $y = |\log_2 |x||$ at $x = \frac{-1}{3}$ is $px + y\ln(q) + \ln(3e) = 0$ then $(p+q)$ is equal to :

(a) 3 (b) $\frac{5}{2}$ (c) 5 (d) $\frac{7}{2}$

32. If the function $f(x) = \sin(\ln x) - 2\cos(\ln x)$ is increasing in the interval (e^{λ}, e^{μ}) then $\sqrt{5}\cos(\mu - \lambda)$ is equal to :

(a) $-\sqrt{5}$ (b) $\frac{-\sqrt{5}}{2}$ (c) 0 (d) $\sqrt{5}$

33. If the curves $x^{2/3} + y^{2/3} = c^{2/3}$ and $(x^2/a^2) + (y^2/b^2) = 1$ touches each other then :

(a) $a + b = c$ (b) $a - b = c$ (c) $a + 2b = c$ (d) $2a - b = c$

34. The value of 'c' in Rolle's theorem for the function $f(x) = \begin{cases} x^2\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ in the interval $[-1, 1]$ is :

(a) $\frac{-1}{2}$ (b) $\frac{1}{4}$

(c) 0 (d) non-existent in the interval

35. Let $f(x) = \begin{cases} e^x, & x \le 0 \\ |1 - x|, & x > 0 \end{cases}$, then which one of the following statement is incorrect?

(a) Continuous and differentiable at $x = 0$

(b) $\lim_{x \to -\infty} f(x) = 0$

(c) One local maxima and one local minima

(d) Decreasing function in (0, 1)

36. If the equation $x^3 - 3x + 1 = 0$ has three real roots x_1, x_2, x_3, where $x_1 < x_2 < x_3$, then the value of $(\{x_1\} + \{x_2\} + \{x_3\})$ is equal to :

[Note: $\{x\}$ denotes the fractional part of x.]

(a) $\frac{3}{2}$ (b) 1 (c) 2 (d) $\frac{5}{2}$

37. Let $f(x) = mx - 1 + \frac{1}{x}$. Then the smallest value of the constant m such that $f(x) \ge 0$ for every $x > 0$ is :

(a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

38. If $f(x) = |||x - 2| - 5| - a|$ then number of integral values of 'a' for which $f(x)$ has exactly 7 critical points, is

(a) 4 (b) 5 (c) 7 (d) 9

39. If $f(x) = (\sqrt{4 - x^2} - 3)^2 + (\sqrt{4 - x^2} + 1)^3$, then the maximum value of $f(x)$, is :

(a) 25 (b) 28 (c) 36 (d) 40

40. If $c \in [0, 1]$ then the minimum value of $\int_0^{4\pi/3} |\sin x - c|\,dx$ occurs when c is equal to :

(a) $\frac{1}{4}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

41. Let $f : R \to R$ be a function defined by $f(x) = 2x^3 - 21x^2 + 78x + 24$. Number of integers in the solutions set of x satisfying the inequality
$f(f(f(x) - 2x^3)) \geq f(f(2x^3 - f(x)))$ is :
(a) 3 (b) 4 (c) 5 (d) 6

42. Consider the three linear equations with respect to X, Y, Z as
$(x)X + Y + 2Z = 0$
$(f(x))X + 3Y + x^2Z = 0$
$(5x)X + 6Y + Z = 0$
having non-trivial solution for X, Y, Z. The curve $f(x)$:
(a) is always increasing (b) is always decreasing
(c) has exactly one critical point (d) has three critical points

43. Let $x_1 = \sqrt[7]{3} + \sqrt[7]{9}$ and $x_2 = \sqrt[7]{5} + \sqrt[7]{7}$, then :
(a) $x_1 = x_2$ (b) $x_1 > x_2$ (c) $x_1 < x_2$ (d) none of these

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 to 3

Consider the cubic $f(x) = 8x^3 + 4ax^2 + 2bx + a$ where $a, b \in R$.

1. For $a = 1$ if $y = f(x)$ is strictly increasing $\forall\, x \in R$ then maximum range of values of b is :
(a) $\left(-\infty, \frac{1}{3}\right]$ (b) $\left(\frac{1}{3}, \infty\right)$ (c) $\left[\frac{1}{3}, \infty\right)$ (d) $(-\infty, \infty)$

2. For $b = 1$, if $y = f(x)$ is non monotonic then the sum of all the integral values of $a \in [1, 100]$, is :
(a) 4950 (b) 5049 (c) 5050 (d) 5047

3. If the sum of the base 2 logarithms of the roots of the cubic $f(x) = 0$ is 5 then the value of 'a' is :
(a) -64 (b) -8 (c) -128 (d) -256

Paragraph for Question Nos. 4 to 6

$$\text{Let } f(x) = \begin{cases} 1-x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } 1 < x \leq 2 \\ (2-x)^2 & \text{if } 2 < x \leq 3 \end{cases} \text{ and } F(x) = \int_0^x f(t)\,dt \text{ then :}$$

4. The function $F(x)$ is :
   (a) increasing in (0, 1) and decreasing in (2, 3)
   (b) decreasing in (0, 1) and increasing in (2, 3)
   (c) decreasing in $(0, 1) \cup (2, 3)$
   (d) increasing in $(0, 1) \cup (2, 3)$

5. Range of $F(x)$ is :
   (a) $\left[0, \frac{1}{2}\right]$ (b) $\left[0, \frac{1}{3}\right]$ (c) $\left[0, \frac{5}{6}\right]$ (d) [0, 1]

6. Area enclosed by the curve $y = F(x)$ and x-axis as x varies from 0 to 3, is :
   (a) $\frac{15}{12}$ (b) $\frac{17}{12}$ (c) $\frac{13}{12}$ (d) $\frac{19}{12}$

Paragraph for Question Nos. 7 to 9

Consider f, g and h be three real-valued differentiable functions defined on R. Let $g(x) = x^3 + g''(1)\,x^2 + (3g'(1) - g''(1) - 1)\,x + 3g'(1)$, $f(x) = x\,g(x) - 12x + 1$ and $f(x) = (h(x))^2$ where $h(0) = 1$.

7. The function $y = f(x)$ has :
   (a) Exactly one local minima and no local maxima
   (b) Exactly one local maxima and no local minima
   (c) Exactly one local maxima and two local minima
   (d) Exactly two local maxima and one local minima

8. Which of the following is/are true for the function $y = g(x)$?
   (a) $g(x)$ monotonically decreases in $\left(-\infty, 2 - \frac{1}{\sqrt{3}}\right) \cup \left(2 + \frac{1}{\sqrt{3}}, \infty\right)$
   (b) $g(x)$ monotonically increases in $\left(2 - \frac{1}{\sqrt{3}}, 2 + \frac{1}{\sqrt{3}}\right)$
   (c) There exists exactly one tangent to $y = g(x)$ which is parallel to the chord joining the points $(1, g(1))$ and $(3, g(3))$
   (d) There exists exactly two distinct Lagrange's mean value in (0, 4) for the function $y = g(x)$.

9. Which one of the following does not hold good for $y = h(x)$?
   (a) Exactly one critical point
   (b) No point of inflection
   (c) Exactly one real zero in (0, 3)
   (d) Exactly one tangent parallel to x-axis

Paragraph for Question Nos. 10 to 12

Consider the functions $f(x)$ and $g(x)$ such that

$$f(x) = \frac{x^3}{2} + 1 - x\int_0^x g(t)\,dt \text{ and } g(x) = x - \int_0^1 f(t)\,dt.$$

Both $f(x)$ and $g(x)$ are defined from $R \to R$.

10. Which one of the following holds good for $f(x)$?

(a) $f(x)$ is bounded

(b) $f(x)$ has exactly one maxima and one minima

(c) $f(x)$ has a maxima but no minima

(d) $f(x)$ has a minima but no maxima

11. Minimum distance between the functions $f(x)$ and $g(x)$ is :

(a) $\frac{4}{3\sqrt{2}}$ (b) $\frac{7}{6\sqrt{2}}$ (c) $\frac{7}{3\sqrt{2}}$ (d) $\frac{8}{3\sqrt{2}}$

12. The function $g(x)$:

(a) is injective but not surjective

(b) cuts the y-axis at $-\frac{3}{2}$

(c) cuts the y-axis at $\frac{3}{2}$

(d) is neither injective nor surjective

Paragraph for Question Nos. 13 to 15

Consider $\Delta(x) = \begin{vmatrix} x^3 - 4x^2 & bx - 2x^2 & 3bx - 8 \\ bx - 2x^2 & 3bx - 8 & x^3 - 4x^2 \\ 3bx - 8 & x^3 - 4x^2 & bx - 2x^2 \end{vmatrix} = f(x) \cdot g(x)$**, where $b \in R^+$ and the equation $f(x) = 0$ has positive real roots. (Leading coefficient of $f(x)$ is 1.)**

13. The value of b is equal to :

(a) 3 (b) 12 (c) -12 (d) -3

14. Number of real roots of the equation $g(x) = 0$, is :

(a) 0 (b) 1 (c) 2 (d) 4

15. If the curve $y = f(x) - 9x^3$ has two horizontal tangents then distance between them, is :

(a) $\frac{8}{4}$ (b) $\frac{45}{2}$ (c) $\frac{27}{2}$ (d) $\frac{81}{8}$

Paragraph for Question Nos. 16 to 18

Let f be a real-valued function on R defined by $f(x) = xe^{-x}$. Suppose a line parallel to the x-axis has distinct intersection points P, Q with the curve $y = f(x)$. Denote the length of the line segment PQ equal to L. Also β is the value of x for which $y = f(x)$ has a local maximum and R be the intersection point of the line $x = \beta$ and the line PQ.

16. If p and q are the abscissa's of the points P and Q respectively, then the ordered pair (p, q) is :

(a) $\left(\frac{L}{e^L - 1}, \frac{e^L}{e^L - 1}\right)$ (b) $\left(\frac{e^L}{e^L - 1}, \frac{L}{e^L - 1}\right)$

(c) $\left(\frac{L}{e^L - 1}, \frac{Le^L}{e^L - 1}\right)$ (d) $\left(\frac{e^L}{e^L - 1}, \frac{e^L}{e^L - 1}\right)$

17. If α denotes the x-coordinate of the point of inflection then $(\alpha - \beta)$ equals :

(a) $e - 1$ (b) $e - 2$

(c) 2 (d) 1

18. Which one of the following relation hold good?

(a) $PR > RQ$

(b) $PR < RQ$

(c) $PR = RQ$

(d) Tricotomy between PR and RQ depends upon the distance between the line PQ and x-axis.

Paragraph for Question Nos. 19 to 21

Let $g(x)$ be a non-constant twice differentiable function defined on R (the set of all real numbers) such that $y = g(x)$ is symmetric about the line $x = 2$ and $g(-2) = g'\left(\frac{1}{2}\right) = g'(1) = 0$.

19. The value of $g(6)$ equals :

(a) $\frac{1}{2}$ (b) 1

(c) 2 (d) 0

20. The minimum number of roots of the equation $g''(x) = 0$ in the interval $(0, 4)$ equals:

(a) 4 (b) 6

(c) 8 (d) 10

21. If $I_1 = \int_{-\pi}^{\pi} g(2+x)\sin x\, dx$ and $I_2 = \int_0^4 \frac{1}{1+e^{g'(x)}} dx$ then which one of the following must hold good?

(a) $I_1 < I_2$

(b) $I_1 > I_2$

(c) $I_1 = I_2$

(d) Nothing definite can be said

Paragraph for Question Nos. 22 to 24

The function $f(x) = \sqrt{ax^3 + bx^2 + cx + d}$ has its non-zero local minimum and local maximum values at – 2 and 2 respectively. Given 'a' is root of the equation $x^2 - x - 6 = 0$.

22. The value of $(a + b + c)$ is equal to :

(a) 16 (b) 18 (c) 20 (d) 22

23. The roots of the equation $ax^2 + bx + c = 0$

(a) are opposite in sign (b) are imaginary

(c) are both positive (d) are both negative.

24. The smallest positive integral value of d is equal to :

(a) 1 (b) 32 (c) 33 (d) 34

Paragraph for Question Nos. 25 to 27

Let $P(x)$ be a polynomial of degree 4, vanishes at $x = -1$ and has local maximum/ local minimum at $x = 1, 2, 3$. Also $\int_{-2}^{2} P(x)\, dx = \frac{-1348}{15}$.

25. The y-intercept of the tangent to the curve $y = P(x)$ at $x = -1$, equals :

(a) 24 (b) – 8

(c) – 96 (d) 34

26. The value of definite integral $\int_{-1}^{1} (P(-x) + P(x))\, dx$ equals :

(a) $\frac{-1244}{15}$ (b) $\frac{-1442}{15}$

(c) $\frac{-2144}{15}$ (d) $\frac{-1424}{15}$

27. The distance between two local minimum points of $y = P(x)$, equals :

(a) 3 (b) 2

(c) 4 (d) 5

Paragraph for Question Nos. 28 to 30

A curve $y = f(x)$ passing through origin and (2, 4). Through a variable point $P(a, b)$ on the curve, lines are drawn parallel to coordinates axes. The ratio of area formed by the curve $y = f(x)$, $x = 0$, $y = b$ to the area formed by the $y = f(x)$, $y = 0$, $x = a$ is equal to 2 : 1.

28. Equation of line touching both the curves $y = f(x)$ and $y^2 = 8x$ is :

(a) $2x + y - 1 = 0$ (b) $2x + y + 1 = 0$
(c) $x + 2y + 1 = 0$ (d) $x + 2y - 1 = 0$

29. Pair of tangents are drawn from the point (3, 0) to $y = f(x)$. The area enclosed by these tangents and $y = f(x)$ is equal to :

(a) 9 (b) 18 (c) 15 (d) 27

30. AB is the chord of curve $y = f(x)$ passing through $\left(0, \frac{1}{4}\right)$. Locus of point of intersection of tangents at A and B is :

(a) $4y + 1 = 0$ (b) $4y - 1 = 0$ (c) $4x + 1 = 0$ (d) $4x - 1 = 0$

Paragraph for Question Nos. 31 to 34

Let $f(x)$ and $g(x)$ be two differentiable functions on R (the set of all real numbers) satisfying $f(x) = \frac{x^3}{2} + 1 - x\int_0^x g(t)\,dt$ and $g(x) = x - \int_0^1 f(t)\,dt$.

31. The value of definite integral $\int_0^1 f(t)\,dt$ lies in the interval :

(a) $\left(0, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 1\right)$ (c) $\left(1, \frac{4}{3}\right)$ (d) $\left(\frac{4}{3}, \frac{5}{3}\right)$

32. Minimum vertical distance between the two curves $f(x)$ and $g(x)$ is :

(a) $\frac{7}{3}$ (b) $\frac{1}{6}$ (c) $\frac{8}{3}$ (d) $\frac{7}{6}$

33. If the distance of the point $P(x_1, y_1)$ on the curve $y = f(x)$ from the curve $y = g(x)$ is least, then x_1 equals :

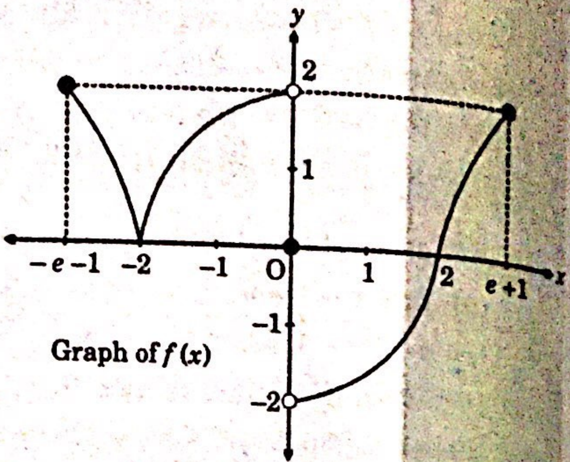
(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{7}{6}$

34. Number of points where $f(|x|)$ is non-derivable, is :

(a) 0 (b) 1 (c) 2 (d) 3

Paragraph for Questions Nos. 35 to 37

Let $f(x) = \begin{cases} 2\ln(-x-1), & -e-1 \le x \le -2 \\ \frac{1}{2}(4-x^2), & -2 < x < 0 \\ 0, & x = 0 \\ \frac{1}{2}(x^2-4), & 0 < x < 2 \\ 2\ln(x-1), & 2 \le x \le e+1 \end{cases}$



Graph of $f(x)$

and graph of $f(x)$ is as shown.

Also, $g(x) = \begin{cases} \min\{f(t) : -e-1 \le t \le x\}, & -e-1 \le x < 0 \\ \max\{f(t) : 0 \le t \le x\}, & 0 \le x \le e+1 \end{cases}$

35. Which one of the following statement does not hold good?

(a) Range of $g(x)$ is $[0, 2]$
(b) $g(x)$ is non-monotonic in $[-2, 2]$
(c) $g(x)$ is a continuous function
(d) $g(x)$ is an odd function

36. If $x = \alpha$ is the point of non-differentiability of $g(x)$ in $(-e-1, e+1)$ then the values of α is :

(a) $-2, 1$
(b) $-2, -1$
(c) $-2, 2$
(d) $-2, 0$

37. If the equation $g(x) = k$ has exactly two distinct solutions in $[-e-1, e+1]$ then the sum of all possible integral values of k is :

(a) 0
(b) 1
(c) 2
(d) 3

Paragraph for Questions Nos. 38 to 40

Consider $f : R \to R$ be a polynomial function whose degree is greater than 1 and atmost 4. Also f is a one-one and onto function whose graph is symmetrical about A (4, 0) and f has horizontal tangent at $x = 4$.

38. The value of definite integral $\int_{-2010}^{2018} f(x)\,dx$ is equal to :

(a) -4
(b) 0
(c) 4
(d) 8

39. The value of $f^{-1}(2) + f^{-1}(-2)$ is equal to :

(a) 0
(b) 2
(c) 4
(d) 8

40. If $f'(10) = 20$, then $f'(-2)$ is equal to :

(a) 4
(b) 8
(c) 20
(d) 32

Paragraph for Question Nos. 41 to 43

Let $f(x)$ be a polynomial of degree 3 satisfying $f(-1) = 10, f(0) = 5$ and $f(x)$ has local maximum at $x = -1$ and $f'(x)$ has local minimum at $x = 1$.

41. The distance between the local maximum and local minimum of the curve $f(x)$, is :
(a) 32 (b) 22 (c) $4\sqrt{65}$ (d) $2\sqrt{55}$

42. Which of the following statement is incorrect?
(a) The slope of normal at point P whose abscissa is 0, on the curve $y = f(x)$ equals $\frac{1}{9}$
(b) There exists no value of $c \in (5, \infty)$ for which $f(c) = 0$
(c) The value of definite integral $\int_0^1 f(x)\,dx$ equals $\frac{-1}{4}$
(d) $f(x)$ has local minimum at $x = -22$

43. Let $g(x) = \int_0^{e^{-x}} \frac{f''(t)}{t^2 - t + 1} dt$, then which of the following statement is correct?
(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
(c) $g'(x)$ does not change sign on $(-\infty, \infty)$
(d) $g'(x)$ change sign on both $(-\infty, 0)$ and $(0, \infty)$

Paragraph for Question Nos. 44 to 45

Let $f(x)$ be a cubic polynomial with leading coefficient unity such that $f(a) = b$ and $f'(a) = f''(a) = 0$. Suppose $g(x) = f(x) - f(a) + (a - x) f'(x) + 3(x - a)^2$ for which conclusion of Rolle's theorem in $[a, b]$ holds at $x = 2$, where $2 \in (a, b)$.

44. The value of $f''(2)$ is equal to :
(a) 2 (b) 3 (c) 4 (d) 6

45. The value of definite integral $\int_a^b f(x)\,dx$ is equal to :
(a) $\frac{123}{64}$ (b) $\frac{213}{64}$ (c) $\frac{321}{64}$ (d) $\frac{481}{64}$

Paragraph for Question Nos. 46 to 48

Let $f(x) = a(x-2)(x-b)$, where $a, b \in R$ and $a \neq 0$.

Also $f(f(x)) = a^3\left(x^2 - (2+b)x + 2b - \frac{2}{a}\right)\left(x^2 - (2+b)x + 2b - \frac{b}{a}\right), a \neq 0,$

has exactly one real zeroes 5.

46. Which of the following is incorrect?

(a) The minimum value of $f(x)$ is 5 and attained at $x = -2$.

(b) The maximum value of $f(x)$ is 2 and attained at $x = 5$.

(c) The minimum value of $f(x)$ is 10 and attained at $x = 0$.

(d) The maximum value of $f(x)$ is 24 and attained at $x = 6$.

47. The slope of the straight line passing through $O(0, 0)$ and tangent to $y = f(x)$, can be :

(a) 4 (b) 2 (c) $\frac{4}{9}$ (d) $\frac{2}{9}$

48. The value of definite integral $\int_2^5 f(x)\,dx$ is equal to :

(a) 4 (b) 8 (c) 10 (d) 14

Paragraph for Question Nos. 49 and 50

Consider a cubic, $f(x) = ax^3 + bx^2 + cx + 4$, $a, b, c \in R$ and $f''\left(\frac{-2}{3}\right) = 0$ and tangent drawn to the graph of the function $y = f(x)$ at $x = \frac{-2}{3}$ is $y = \frac{5x}{3} + \frac{100}{27}$.

49. The value of $(a + b + c)$ is equal to :

(a) 4 (b) 6 (c) 7 (d) 10

50. If g is the inverse of f then $\frac{d}{dx}(g(x) \cdot f\{g(x)\})$ at $x = 4$ is equal to :

(a) $\frac{1}{4}$ (b) $\frac{4}{7}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

Paragraph for Question Nos. 51 and 52

Consider the polynomial $p(x) = 9x^8 - 18x^5 - 20x^3 + 15$

51. $p(x) = 0$ has :

(a) exactly one real root in $[0, 3^{1/3}]$

(b) exactly two real roots in $[0, 5^{1/5}]$

(c) at least two real roots in $[0, 3^{1/3}]$

(d) no real roots in $[3^{1/5}, 5^{1/5}]$

52. If the graph of $y = p(x)$ represents the rate of change of slope of tangent at any 'x' to the curve $y = f(x)$ then number of horizontal tangents to the curve $y = f(x)$ will be [Given that $f'(1) = 8$] :

(a) 0 (b) 1 (c) 2 (d) 3

Paragraph for Question Nos. 53 to 55

Let $f(x)$ be a real valued function defined on positive real numbers. The tangent lines drawn to the graph of $y = f(x)$ always intersect the y-axis 1 unit lower than where they meet the function. Also $f(1) = 0$.

53. Range of the function is :

(a) $(-\infty, \infty)$ (b) $(0, \infty)$ (c) $[1, \infty)$ (d) $[-1, 1]$

54. If $xf(|x|) = k$ has exactly one distinct solution then true set of values of k is equal to :

(a) $\left(0, \frac{1}{2}\right)$ (b) $\left(-\infty, \frac{-1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$

(c) $\left(\frac{-1}{e}, \frac{1}{e}\right)$ (d) $\left(\frac{-1}{e}, 0\right)$

55. The longest interval where the function $g(x) = \frac{x}{f(x)}$ is decreasing is denoted by J.

Number of integral values in the interval J is :

(a) 0 (b) 1 (c) 2 (d) 3

Paragraph for Question Nos. 56 to 58

Let $f(x)$ be function defined on the set of real numbers to real numbers, such it is continuous on R.

56. If $f(x)$ is also differentiable on R such that $f'(x) > f(x) \forall x \in R$ and $f(x_0) = 0$ then :

(a) $f(x) < 0 \forall x > x_0$ (b) $f(x) \geq 0 \forall x > x_0$

(c) $f(x) > 0 \forall x > x_0$ (d) $f(x) \leq 0 \forall x > x_0$

57. The equation $k \cdot e^x = 5 + x + \frac{x^2}{2}$ where k is a positive constant has :

(a) exactly two roots (b) exactly one root

(c) no real root (d) many roots

58. If $f(x) = x^3 - (9 - a)x^2 + 3(9 - a^2)x + 7$ has points of extrema which are of opposite sign, then the values of parameter a :

(a) $(0, \infty) \cup \left(-\infty, -\frac{4}{9}\right)$ (b) $(-3, 3) \cup (4, \infty)$

(c) $\left(-3, -\frac{4}{9}\right) \cup (0, 3)$ (d) $(-\infty, -3) \cup (3, \infty)$

EXERCISE - 3

More Than One Correct Answers

1. Consider $f(x) = e^x \sec x - \sqrt{2} \cos x + x, x \in \left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$

(a) minimum value of $f(x)$ is $\left(\sqrt{2}e^{\frac{-\pi}{4}} - 1 - \frac{\pi}{4}\right)$

(b) minimum value of $f(x)$ is $\left(2e^{\frac{-\pi}{3}} - \frac{1}{\sqrt{2}} - \frac{\pi}{3}\right)$

(c) $f'\left(\frac{\pi}{3}\right) \geq f'(x)\ \forall\ x \in \left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$

(d) $f'\left(\frac{\pi}{3}\right) \leq f'(x)\ \forall\ x \in \left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$

2. If $\phi(x) = f(x) + f(2a - x)$ and $f''(x) > 0$, $a > 0$, $0 \leq x \leq 2a$ then :

(a) $\phi(x)$ increases in $(a, 2a)$ (b) $\phi(x)$ increases in $(0, a)$

(c) $\phi(x)$ decreases in $(0, a)$ (d) $\phi(x)$ decreases in $(a, 2a)$

3. A function f is defined by $f(x) = \int_0^{\pi} \cos t \cos(x - t)\, dt$, $0 \leq x \leq 2\pi$ then which of the following hold(s) good?

(a) $f(x)$ is continuous but not differentiable in $(0, 2\pi)$

(b) Maximum value of f is π

(c) There exists atleast one $c \in (0, 2\pi)$ $s.t.$ $f'(c) = 0$

(d) Minimum value of f is $-\frac{\pi}{2}$

4. If $\lim_{x\to a} f(x) = \lim_{x\to a} [f(x)]$ (a is a finite quantity), where $[\cdot]$ denotes greatest integer function and $f(x)$ is a non constant continuous function, then :

(a) $\lim_{x\to a} f(x)$ is an integer (b) $\lim_{x\to a} f(x)$ need not be an integer

(c) $f(x)$ has a local minimum at $x = a$ (d) $f(x)$ has a local maximum at $x = a$

5. Which of the following is(are) correct?

(a) $3^{\pi} > \pi^3$ (b) $\sin x > x - \frac{x^3}{6}\ \forall\ x > 0$

(c) $2 > \left(\frac{5}{2}\right)^{\frac{4}{5}}$ (d) $10^{11} > 11^{10}$

6. Let f be real-valued function on R defined as $f(x) = x^4(1 - x)^2$, then which of the following statement(s) is(are) correct ?

(a) $f'(c) = 0$ for some $c \in (0, 1)$

(b) $f''(x)$ vanishes exactly twice in R

(c) $f(x)$ is an even function

(d) Monotonic increasing in $\left(0, \frac{2}{3}\right) \cup (1, \infty)$

7. Let $f(x) = \int_1^x \frac{3^t}{1+t^2}\, dt$, $x > 0$ then :

(a) for $0 < \alpha < \beta$, $f(\alpha) < f(\beta)$ (b) for $0 < \alpha < \beta$, $f(\alpha) > f(\beta)$

(c) for all $x > 0$, $f(x) + \frac{\pi}{4} < \tan^{-1} x$ (d) for all $x > 0$, $f(x) + \frac{\pi}{4} > \tan^{-1} x$

8. Let $f(x)=\dfrac{(x-1)^2 . e^x}{(1+x^2)^2}$, then which of the following statement(s) is(are) correct?

(a) $f(x)$ is strictly increasing in $(-\infty, -1)$

(b) $f(x)$ is strictly decreasing in $(1, \infty)$

(c) $f(x)$ has two points of local extremum

(d) $f(x)$ has a point of local minimum at some $x \in (-1, 0)$

9. If f is an odd continuous function in $[-1, 1]$ and differentiable in $(-1, 1)$, then which of the following statement(s) is(are) correct?

(a) $f'(a) = f(1)$ for some $a \in (-1, 0)$

(b) $f'(b) = f(1)$ for some $b \in (0, 1)$

(c) $n(f(\alpha))^{n-1} f'(\alpha) = (f(1))^n$ for some $\alpha \in (-1, 0)$ and $\forall\, n \in N$

(d) $n(f(\beta))^{n-1} f'(\beta) = (f(1))^n$ for some $\beta \in (0, 1)$ and $n \in N$

10. Let $f(x)=\begin{cases}(1-x)^{\alpha}\cdot x\cdot(1-\cos(2\pi x)), & 0<x\le 1\\ 0, & x=0\end{cases}$. If Rolle's theorem is applicable to $f(x)$ for $x \in (0, 1)$, then α can be :

(a) -2 (b) -1 (c) $\dfrac{1}{2}$ (d) 1

11. Let $f(x)=\int_0^x e^{-t^2}(t-5)(t^2-7t+12)\,dt$ for all $x \in (0, \infty)$, then :

(a) f has a local maximum at $x = 4$ and a local minimum at $x = 3$

(b) f is decreasing on $(3, 4) \cup (5, \infty)$ and increasing on $(0, 3) \cup (4, 5)$

(c) There exists atleast two $c_1, c_2 \in (0, \infty)$ such that $f''(c_1) = 0$ and $f''(c_2) = 0$

(d) There exists some $c \in (0, \infty)$ such that $f'''(c) = 0$

12. For which of the following functions **Rolle's Theorem** is applicable?

(a) $f(x)=\dfrac{x}{2}+\dfrac{2}{x}, x \in [1, 4]$ (b) $f(x) = x + 1 - x^{\frac{3}{2}}, x \in [0, 1]$

(c) $f(x) = |x+1|^3, x \in [-2, 0]$ (d) $f(x) = \operatorname{sgn}(x) + \operatorname{sgn}(-x), x \in \left[\dfrac{-5}{2}, \dfrac{1}{2}\right]$

[Note : sgn (x) denotes signum function of x.]

13. If $f(x) = \min.(1, \cos x, 1 - \sin x), -\pi \le x \le \pi$, then :

(a) $f(x)$ is not differentiable at $x = 0$

(b) $f(x)$ has local maximum at $x = 0$

(c) $f(x)$ is differentiable at $x = \dfrac{\pi}{4}$

(d) $f(x)$ is continuous and bounded in $x \in [-\pi, \pi]$

14. Let $f(x) = \int_{-2}^{x} (t^2 - t + 2)(t^2 - t - 2)(t^2 - t - 6)(t^2 - t - 12)\, dt$ for all $x \in R$, then which of the following statements is (are) correct?

(a) The equation of normal to $f(x)$ at $x = -2$ is $x = -2$

(b) $f(x)$ increases in $(-3, -2) \cup (-1, 2) \cup (3, 4)$

(c) $f(x)$ decreases in $(-\infty, -3) \cup (-2, -1) \cup (2, 3) \cup (4, \infty)$

(d) The sum of values of x at which $f(x)$ has local maximum equals -1

15. Let the function $f(x)$ be thrice differentiable and satisfies $f(f(x)) = 1 - x$ for all $x \in [0, 1]$. If $J = \int_0^1 f(x)\, dx$ and $f''\left(\frac{4}{5}\right) = 0$, then which of the following is (are) true?

(a) $f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) = 1$

(b) $J = \frac{1}{2}$

(c) $f''(x) = 0$ has atleast one root in $x \in \left(\frac{1}{4}, \frac{3}{4}\right)$

(d) $f'''(x) = 0$ has at least one root in $x \in \left(\frac{1}{2}, \frac{4}{5}\right)$

16. Let $f(x) = \begin{cases} \sin^{-1}(\sin x), & x > 0 \\ \frac{\pi}{2}, & x = 0 \\ \cos^{-1}(\cos x) & x < 0 \end{cases}$, then :

(a) $f(x)$ has local maxima at $x = 0$

(b) $f(x)$ is continuous for all $x \in R$

(c) $f(x)$ has maximum value π

(d) $f(x)$ has minimum value 0

17. If $f(x) = |x^2 - 5|x| + 6|$ then :

(a) $f(x)$ is non-differentiable at 5 points

(b) $f(x)$ is non-differentiable at 4 points

(c) $f(x)$ has local maxima at $x = 0$

(d) $f(x) = k$ has 8 solutions for $k \in \left(0, \frac{1}{4}\right)$

18. Let $f(x) = \cot x - \tan x - 2\tan 2x - 4\tan 4x - 8\cot 8x$, $x \neq \frac{n\pi}{8}$, $n \in I$ and $g(x) = x^3 + 6x - 1$. One more function $h(x)$ is defined as $h : R - \left\{\frac{n\pi}{8}, n \in I\right\} \to R$, $h(x) = f(x) + g(x)$ then identify the correct statement(s).

(a) $h''\left(\frac{\pi}{24}\right) = \frac{\pi}{4}$

(b) $h(x)$ is odd function

(c) $h(x)$ is increasing in the domain.

(d) If the equation $h(x) = \lambda$ has a solution in $(0, 3)$ then number of integral values of λ is 7.

19. Let $f(x) = \cos^2 x \cdot e^{\tan x}, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ then :

(a) $f'(x)$ has a point of local minima at $x = \frac{\pi}{4}$

(b) $f'(x)$ has a point of local maxima in $\left(\frac{-\pi}{4}, 0\right)$

(c) $f'(x)$ has exactly two points of local maxima / minima in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(d) $f''(x) = 0$ has no root in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

20. Let $f: R \to R$ be defined by $f(x) = \begin{cases} 2x^2 \ln|x| - 5x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Which of the following statement(s) is/are correct?

(a) $f(x)$ has exactly one local maximum and two local minimum points

(b) $f(x)$ is strictly increasing in $(10, \infty)$

(c) Absolute minimum value of $f(x)$ exist but absolute maximum value of $f(x)$ does not exist

(d) In the interval $x \in (0, \infty)$, $g(x) = k$ has two distinct roots

21. If the tangent at a point P_1 (other than (0,0)) on the curve $ax^3 - y + b = 0$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 and so on. If the abscissae of $P_1, P_2, P_3, \ldots P_n$ form a G.P. then (a, b) may be :

(a) (1, 0) (b) (2, 7) (c) (3, 5) (d) (4, 9)

EXERCISE - 4

Match the Columns Type

1. Let $f(x) = (x-1)(x-2)(x-3) \ldots\ldots\ldots (x-n), n \in N$ and $\int \frac{f(x) f''(x) - [f'(x)]^2}{f^2(x)} dx = g(x) + C$, where C is arbitrary constant.

Column I	Column II
(a) If $f'(n) = 5040$, then n is divisible by	(p) 4
(b) If $g(x)$ is discontinuous at 9 points, $\forall x \in R$ then n is greater than	(q) 6
(c) If $g(x) = 5$ has 8 solutions, then n may be equal to	(r) 8
(d) If the number of roots of equation $f'(x) = 0$, be $(n-5)^2 (n-1)$, $(n > 1)$ then possible values of n is/are	(s) 9

2. Column I **Column II**

(a) Let $f(x) = x^3 + ax^2 + ax + 1$ has local extrema at $x = \alpha, \beta$ where $\alpha < \beta$. If $f(\alpha) + f(\beta) = 2$ then the value of 'a' equals — (p) 9/2

(b) Value of $p + q$ for which $f(x) = x^3 + px^2 + qx + r$, where $p, q, r \in R$ is monotonically decreasing in largest possible interval $(-5/3, -1)$, is — (q) 11/2

(c) If the equation $e^{2x} = k\sqrt{x}$ has exactly 2 distinct solutions then 'k' can be equal to — (r) 7

(s) 9

3. Column I **Column II**

(a) Let $f(x) = \begin{cases} x^{3/5} & \text{if } x \le 1 \\ -(x-2)^3 & \text{if } x > 1 \end{cases}$ then the number of critical points on the graph of the function is — (p) 5

(b) Number of real solution of the equation, $\log_2^2 x + (x-1)\log_2 x = 6 - 2x$, is — (q) 4

(c) The number of values of c such that the straight line $3x + 4y = c$ touches the curve $\frac{x^4}{2} = x + y$ is — (r) 3

(d) If $f(x) = \int_x^{x^2} (t-1)\,dt$, $1 \le x \le 2$, then global maximum value of $f(x)$ is — (s) 2

(t) 1

4. Column I **Column II**

(a) If $\prod b$ denotes the product of all possible values of b and $\sum b$ denotes the sum of all possible values of b, for which a line tangent to the graph of $f(x) = x^3 - \frac{11x}{3}$ at the point $M(b, f(b))$ passes through the point $N(3, 0)$, then $\sum b - \prod b$ has the value equal to — (p) 1

(b) The real value of a for which the integral $\int_{a-1}^{a+1} e^{-(x-1)^2}\,dx$ attains its maximum value is equal to — (q) 4

(c) Line l is tangent to the curve $y = e^x$ and is parallel to the line $x - 4y = 1$. If x-intercept of the line is $-\ln(ke)$, $k \in N$ then k is equal to — (r) 6

(s) 10

5. Consider $f(x) = |\ln|x|| - kx^2$, $x \neq 0$. Match the column I with the value of k in column II.

Column I	Column II
(a) $f(x) = 0$ has two distinct solutions	(p) $k = 0$
(b) $f(x) = 0$ has four distinct solutions	(q) $k = \frac{1}{2e}$
(c) $f(x) = 0$ has six distinct solutions	(r) $k \in \left(\frac{1}{2e}, \infty\right)$
(d) $f(x) = 0$ has no solution	(s) $k \in (-\infty, 0)$
	(t) $k \in \left(0, \frac{1}{2e}\right)$

EXERCISE - 5

Integer Answer Type

1. Let $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + 2$ and $g'(x) = (x^2 - 9)(x^2 - 4x + 3)(x^2 - 3x + 2)(x^2 - 2x - 3)$. If n_1, n_2 and n_3 denote number of points of local minima, number of points of local maxima and number of points of inflection of the function $f[g(x)]$ then find the value of $(n_1 + n_2 + n_3)$.

2. If all the real values of m for which the function $f(x) = \frac{x^3}{3} - (m-3)\frac{x^2}{2} + mx - 2013$ is strictly increasing in $x \in [0, \infty)$ is $[0, k]$, then find the value of k.

3. Let f be a continuous function and satisfies $f'(x) > 0$ on $(-\infty, \infty)$ and the value of $f''(x)\ \forall\ x \in (0, \infty)$ is equal to minimum value of $\min_{x \in R} \{e^{-|x|} + 2, |x| + 2\}$.

If $L = \lim_{x \to \infty} \frac{3x^2 - \frac{3}{x^2+1} - 4f'(x)}{f(x)}$ then find the value of $[L^2]$.

[**Note :** $[k]$ denotes greatest integer less than or equal to k.]

4. If $a^2 + b^2 = 1$ and u is the minimum value of $\frac{b+1}{a+b-2}$ then find the value of u^2.

5. Let $f(x)$ be a continuous and differentiable function satisfying the following conditions

(a) $\prod_{r=1}^{6} f(r) < 0, \prod_{r=1}^{3} f(2r) < 0, \prod_{r=1}^{2} f(3r) < 0$

(b) $f(6) > 0$, and

(c) $f(x)$ is monotonic in $(n, n+1)$, $n \in I$

Let A denotes the set consisting of number of distinct possible roots of $f(x) = 0$ in $x \in (1, 6)$. Find the sum of all the elements of set A.

6. A circle of radius 1 unit touches the positive X-axis and positive Y-axis at P and Q respectively. A variable line L passing through the origin intersects the circle in two points M and N. If m is the slope of the line L for which the area of the triangle MNQ is maximum, then find the value of $2010\,(m^2)$.

7. For constant number 'a', consider the function $f(x) = ax + \cos 2x + \sin x + \cos x$ on R (the set of real numbers) such that $f(u) < f(v)$ for $u < v$. If the range of 'a' for any real numbers u, v is $\left[\frac{m}{n}, \infty\right)$, then find the minimum value of $(m + n)$.

8. Let $f(x) = \frac{\pi}{2} + \left|\operatorname{sgn}\left(\tan^{-1}\left(\frac{x}{1+x^2}\right)\right)\right| \tan^{-1} x$, where sgn (y) denotes signum function of y, and $g(x)$ is the inverse of $f(x)$. If the complete set of values of k for which the equation $2g(x) + k(\pi - 2x) = 0$ has three distinct solutions is (a, ∞) then find 'a'.

9. Let A (1, – 1), B (4, – 2) and C (9, 3) be the vertices of the triangle ABC. A parallelogram $AFDE$ is drawn with vertices D, E and F on the line segments BC, CA and AB respectively. Find the maximum area of parallelogram $AFDE$.

10. Let $f : [0, \infty) \to R$ be a continuous, strictly increasing function such that $f^3(x) = \int_0^x tf^2(t)\, dt$. If a normal is drawn to the curve $y = f(x)$ with gradient $\frac{-1}{2}$, then find the intercept made by it on the y-axis.

11. A polynomial function $P(x)$ of degree 5 with leading coefficient one, increases in the interval $(-\infty, 1)$ and $(3, \infty)$ and decreases in the interval $(1, 3)$. Given that $P(0) = 4$ and $P'(2) = 0$. Find the value $P'(6)$.

12. Let f be a differentiable function on R and satisfying $f(x) = -(x^2 - x + 1)\, e^x + \int_0^x e^{x-y} . f'(y)\, dy$. If $f(1) + f'(1) + f''(1) = ke$, where $k \in N$, then find k.

13. Let a_n $(n \geq 1)$ be the value of x for which $\int_x^{2x} e^{-t^n}\, dt$ $(x > 0)$ is maximum. If $L = \lim_{n\to\infty} \ln(a_n)$ then find the value of e^{-L}.

14. Find the absolute value of k $(k \in R)$ for which $f(k) = \int_{-\pi}^{\pi} (x + 1 + k\sin x)^2\, dx$ is minimum.

15. Let $f(x) = \begin{cases} (x+2)^3, & -3 < x \leq -1 \\ x^{\frac{2}{3}}, & -1 < x < 2 \end{cases}$ and $g(x) = \int_{-3}^{x} f(t)\, dt, -3 < x < 2$. Find the number of extremum points of $g'(x)$.

16. Let $f(x) = \int_0^x 3^t (3^t - 4)(x - t)\, dt$ $(x \geq 0)$. If $x = a$ is the point where $f(x)$ attains its local minimum value then find the value of 3^a.

17. Find the length of the shortest path that begins at the point (2, 5), touches the x-axis and then ends at a point on the circle $x^2 + y^2 + 12x - 20y + 120 = 0$.

18. If the exhaustive set of all possible values of c such that $f(x) = e^{2x} - (c + 1) e^x + 2x + \cos 2 + \sin 1$, is monotonically increasing for all $x \in R$, is $(-\infty, \lambda]$, then find the value of λ.

19. Let $p(x)$ be fifth degree polynomial such that $p(x) + 1$ is divisible by $(x - 1)^3$ and $p(x) - 1$ is divisible by $(x + 1)^3$. Then find the value of definite integral $\int_{-10}^{10} p(x)\, dx$.

20. Let $P(x_0, y_0)$ be a point on the curve $C : (x^2 - 11)(y + 1) + 4 = 0$ where $x_0, y_0 \in N$. If area of the triangle formed by the normal drawn to the curve 'C' at P and the co-ordinate axes is $\left(\frac{a}{b}\right)$, $a, b \in N$ then find the least value of $(a - 6b)$.

21. Let $M = \left(p, \frac{4}{3 - p} - 1\right)$ be a variable point which moves in $x - y$ plane. If $d = \sqrt{a} - \sqrt{b}$, $a, b \in N$ is the least distance of the point M to the circle $(x - 3)^2 + (y + 1)^2 = 1$, then find the value of $(a - b)$.

22. Let f be a twice differentiable function defined in $[-3, 3]$ such that $f(0) = -4$, $f'(3) = 0$, $f'(-3) = 12$ and $f''(x) \geq -2\ \forall\ x \in [-3, 3]$. If $g(x) = \int_0^x f(t)\, dt$ then find maximum value of $g(x)$.

23. Let $f(x) = \int_x^{x+\frac{\pi}{3}} |\sin \theta|\, d\theta, 0 \leq x \leq \pi$. If m and M are minimum and maximum values of $f(x)$ and $m + M = \sqrt{p} - \sqrt{q}$ where $p, q \in N$, then find the value of $(p + q)$.

24. If $f(x) = a\ |\cos x| + b\ |\sin x|$ $(a, b \in R)$ has a local minimum at $x = \frac{-\pi}{3}$ and satisfies $\int_{-\pi/2}^{\pi/2} (f(x))^2\, dx = 2$. Find the values of a and b and hence find $\frac{b^2}{a^2}$.

25. If the maximum value of the expression $y = \frac{x^4 - x^2}{x^6 + 2x^3 - 1}$ for $x > 1$, is $\frac{p}{q}$ and it occurs at $x = \frac{a + \sqrt{b}}{c}$ where p and q are in their lowest term and a, b, c are pairwise relatively prime positive numbers, find the value of $(a + b + c + p + q)$.

26. For $a > 0$, find the minimum value of the integral $\int_0^{1/a}(a^3 + 4x - a^5x^2)e^{ax}dx$.

27. Let $P(x)$ be a polynomial of degree 3 such that $P(x) + 2$ is divisible by $(x-2)^2$ and $P(x) - 2$ is divisible by $(x+2)^2$. If the value of $P(-3)$ is equal to $\frac{a}{b}$. (where $a, b \in N$) then find the minimum value of $(a + b)$.

28. Find the sum of all integral values of a such that $a(x^2+x-1) \leq (x^2+x+1)^2 \; \forall x \in R$.

29. Let f be a real function defined on R (the set of real numbers) such that $f'(x) = 100(x-1)(x-2)^2(x-3)^3 \ldots\ldots\ldots\ldots (x-100)^{100}$, for all $x \in R$. If g is a function defined on R such that $\int_a^x e^{f(t)}dt = \int_0^x g(x-t)\, dt + 2x + 3$, then find the sum of all the values of x for which $g(x)$ has a local extremum.

30. Let $f(x)$ be a differentiable function on R defined by $f(x) = 5-(x+1)^2$, and A be the point of intersection where the tangent line drawn to the graph of $y = f(x)$ at the point $P(x, f(x))$ intersects with x-axis and B be the intersection point where the tangent line intersects with y-axis. If $S(x)$ denotes the area of ΔOAB where O is the origin, then the minimum value of $S(x)$ in the interval $(0, 1)$ is equal to $\frac{p}{q}$ (where p and q are in their lowest form). Find the value of $(p + q)$.

31. Let the function $f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$ $(a \neq 0)$ touches the line $y = g(x) = x + p$ at $x = 1, 2$ and 3, and $h(x) = f(x) - g(x)$. If $\int_1^3 h(x)\,dx = \frac{32}{105}$, find the value of a.

32. Consider an equation with x as a variable $7\sin 3x - 2\sin 9x = \sec^2\theta + 4\operatorname{cosec}^2\theta$, then find the value of $\frac{15}{\pi}$[(minimum positive root) – (maximum negative root)].

33. Let $g(a) = \int_0^{\pi/2} |\sin 2x - a\cos x|\, dx$, $0 \leq a \leq 1$. If m and M is minimum and maximum value of $g(a)\ \forall a \in [0, 1]$ then find the value of $2(M + m)$.

34. If $9 + f''(x) + f'(x) = x^2 + f^2(x)$, where $f(x)$ is twice differentiable function such that $f''(x) \neq 0 \forall x \in R$ and let P be the point of maxima of $f(x)$ then find the number of tangents which can be drawn from P to the circle $x^2 + y^2 = 9$.

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (b)	2. (d)	3. (a)	4. (c)	5. (c)	6. (d)	7. (d)	8. (a)	9. (c)	10. (a)
11. (b)	12. (d)	13. (a)	14. (c)	15. (b)	16. (a)	17. (a)	18. (c)	19. (c)	20. (d)
21. (b)	22. (c)	23. (a)	24. (d)	25. (c)	26. (b)	27. (a)	28. (a)	29. (c)	30. (c)
31. (d)	32. (a)	33. (a)	34. (c)	35. (a)	36. (b)	37. (b)	38. (a)	39. (b)	40. (c)
41. (c)	42. (a)	43. (c)							

EXERCISE 2 : Linked Comprehension Type

1. (c)	2. (b)	3. (d)	4. (d)	5. (c)	6. (b)	7. (c)	8. (d)	9. (c)	10. (d)
11. (c)	12. (b)	13. (a)	14. (a)	15. (c)	16. (c)	17. (d)	18. (b)	19. (d)	20. (a)
21. (a)	22. (d)	23. (a)	24. (c)	25. (c)	26. (d)	27. (b)	28. (b)	29. (b)	30. (a)
31. (d)	32. (a)	33. (b)	34. (a)	35. (d)	36. (c)	37. (d)	38. (b)	39. (d)	40. (c)
41. (c)	42. (d)	43. (b)	44. (d)	45. (c)	46. (a, c, d)		47. (a, c)	48. (a)	49. (b)
50. (c)	51. (c)	52. (d)	53. (a)	54. (b)	55. (b)	56. (c)	57. (b)	58. (d)	

EXERCISE 3 : More Than One Correct Answers

1. (a, c)	2. (a, c)	3. (c, d)	4. (a, c)	5. (a, b, d)
6. (a, d)	7. (a)	8. (a, c)	9. (a, b, d)	10. (c, d)
11. (a, c, d)	12. (a, b, c, d)	13. (a, b, c, d)	14. (a, d)	15. (a, b, c, d)
16. (a, c)	17. (a, c, d)	18. (a, c)	19. (a, b, c)	20. (a, b, c)
21. (a, b, c, d)				

EXERCISE 4 : Match the Columns Type

1. **(a)** (p) (r), **(b)** (p) (q) (r), **(c)** (r) (s), **(d)** (p) (q)
2. **(a)** (p), **(b)** (s), **(c)** (q) (r) (s)
3. **(a)** (r), **(b)** (s), **(c)** (t), **(d)** (q)
4. **(a)** (s), **(b)** (p), **(c)** (q)
5. **(a)** (r) (p), **(b)** (q), **(c)** (t), **(d)** (s)

EXERCISE 5 : Integer Answer Type

1. 5	2. 9	3. 9	4. 9	5. 10
6. 670	7. 25	8. 1	9. 5	10. 9
11. 1200	12. 9	13. 2	14. 2	15. 2
16. 7	17. 13	18. 3	19. 0	20. 3
21. 7	22. 48	23. 12	24. 3	25. 15
26. 4	27. 17	28. 36	29. 2500	30. 107
31. 2	32. 10	33. 3	34. 0	

Chapter 8

Differential Equations

KEY CONCEPTS

I. DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

DEFINITIONS :

(a) An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **Differential Equation.**

(b) A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only and it is said to be **Partial** if there are two or more independent variables. We are concerned with ordinary differential equations only. *e.g.* $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ is a partial differential equation.

(c) Finding the unknown function is called **Solving or Integrating** the differential equation. The solution of the differential equation is also called its **Primitive,** because the differential equation can be regarded as a relation derived from it.

(d) The order of a differential equation is the order of the highest differential coefficient occuring in it.

(e) The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occuring in it, after it has been expressed in a form free from radicals and fractions so far as derivatives are concerned, thus the differential equation :

$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p+\phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q+\ldots\ldots=0$ is order m and degree p.

Note that in the differential equation $e^{y'''} - xy'' + y = 0$ order is three but degree doesn't apply.

2. FORMATION OF A DIFFERENTIAL EQUATION :

If an equation in independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows :

(a) Differentiate the given equation *w.r.t.* the independent variable (say x) as many times as the number of arbitrary constants in it.

(b) Eliminate the arbitrary constants.

The eliminant is the required differential equation. Consider forming a differential equation for $y^2 = 4a(x + b)$ where a and b are arbitary constant.

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

3. GENERAL AND PARTICULAR SOLUTIONS :

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **General Solution (Complete Integral or Complete Primitive).** A solution obtainable from the general solution by giving particular values to the constants is called a **Particular Solution.**

Note that the general solution of a differential equation of the n^{th} order contains 'n' and only 'n' independent arbitrary constants. The arbitrary constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation $y = Ae^{x+B}$ are not independent since the equation can be written as $y = Ae^B . e^x = Ce^x$. Similarly the solution $y = A \sin x + B \cos(x + C)$ appears to contain three arbitrary constants, but they are really equivalent to two only.

4. ELEMENTARY TYPES OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS :

TYPE–1 :

Variables Separable : If the differential equation can be expressed as;

$f(x)\,dx + g(y)\,dy = 0$ then this is said to be variables separable type.

A general solution of this is given by $\int f(x)\,dx + \int g(y)\,dy = c$; where c is the arbitrary constant. Consider the example $(dy/dx) = e^{x-y} + x^2 . e^{-y}$.

> **Note :** Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials.
>
> If $x = r\cos\theta$; $y = r\sin\theta$ then,
>
> **1.** $x\,dx + y\,dy = r\,dr$
>
> **2.** $dx^2 + dy^2 = dr^2 + r^2\,d\theta^2$
>
> **3.** $x\,dy - y\,dx = r^2\,d\theta$
>
> If $x = r\sec\theta$ and $y = r\tan\theta$ then $x\,dx - y\,dy = r\,dr$ and $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

TYPE–2 :

$\dfrac{dy}{dx} = f(ax + by + c), b \neq 0.$

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Consider the example $(x + y)^2 \dfrac{dy}{dx} = a^2$.

TYPE–3 :

Homogeneous Equations : A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{\phi(x, y)}$,

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x and y, and of the same degree, is called **Homogeneous**. This equation may also be reduced to the form $\dfrac{dy}{dx} = g\left(\dfrac{x}{y}\right)$ and is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable. Consider $\dfrac{dy}{dx} + \dfrac{y(x + y)}{x^2} = 0.$

TYPE–4 :

Equations Reducible to the Homogeneous Form : If $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$; where $a_1 b_2 - a_2 b_1 \neq 0$, *i.e.* $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$ then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in **Type – 3.** If :

(a) $a_1 b_2 - a_2 b_1 = 0$, then a substitution $u = a_1 x + b_1 y$ transforms the differential equation to an equation with variables separable, and

(b) $b_1 + a_2 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $x\,dy + y\,dx$ and integrating term by term yields the result easily.

Consider $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1}$; $\frac{dy}{dx} = \frac{2x + 3y - 1}{4x + 6y - 5}$ and $\frac{dy}{dx} = \frac{2x - y + 1}{6x - 5y + 4}$

(c) In an equation of the form : $y\,f(xy)\,dx + xg(xy)\,dy = 0$ the variables can be separated by the substitution $xy = v$.

Note :

1. The function $f(x, y)$ is said to be a homogeneous function of degree n if for any real number t ($\neq 0$), we have $f(tx, ty) = t^n f(x, y)$.
   For *e.g.* $f(x, y) = ax^{2/3} + hx^{1/3} \cdot y^{1/3} + by^{2/3}$ is a homogeneous function of degree $\frac{2}{3}$.

2. A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is homogeneous if $f(x, y)$ is a homogeneous function of degree zero *i.e.* $f(tx, ty) = t^{\circ} f(x, y) = f(x, y)$. The function f does not depend on x and y separately but only on their ratio $\frac{y}{x}$ or $\frac{x}{y}$.

Linear Differential Equations : A differential equation is said to be linear if the dependent variable and its differential coefficients occur in the first degree only and are not multiplied together.

The nth order linear differential equation is of the form;

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_n(x) \cdot y = \phi(x)$. Where $a_0(x), a_1(x), \ldots\ldots a_n(x)$ are called the coefficients of the differential equation.

Note : A linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. *e.g.* the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^2=0$ is not linear, though its degree is 1.

TYPE–5 :

Linear Differential Equations of First Order : The most general form of a linear differential equations of first order is $\frac{dy}{dx}+Py=Q$, where P and Q are functions of x.

To solve such an equation multiply both sides by $e^{\int P\,dx}$.

Note :

1. The factor $e^{\int P\,dx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x and y, is called integrating factor of the differential equation popularly abbreviated as $I.F.$
2. It is very important to remember that on multiplying by the integrating factor, the left hand side becomes the derivative of the product of y and the $I.F.$
3. Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. *e.g.*, the equation; $(x+y+1)\frac{dy}{dx}=y^2+3$ can be written as $(y^2+3)\frac{dx}{dy}=x+y+1$ which is a linear differential equation.

TYPE–6 :

Equations Reducible to Linear Form : The equation $\frac{dy}{dx}+py=Q\,.\,y^n$ where P and Q functions of x, is reducible to the linear form by dividing it by y^n and then substituting $y^{-n+1}=Z$. Its solution can be obtained as in **Type–5**. Consider the example $(x^3y^2+xy)\,dx=dy$.

The equation $\frac{dy}{dx}+Py=Q\,.\,y^n$ is called **Bernouli's Equation.**

5. TRAJECTORIES :

Suppose we are given the family of plane curves.

$\Phi(x, y, a)=0$

depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an ***isogonal trajectory*** of that family; if in particular $\alpha = \frac{\pi}{2}$, then it is called an ***orthogonal trajectory***.

Orthogonal trajectories : We set up the differential equation of the given family of curves. Let it be of the form

$$F(x, y, y') = 0$$

The differential equation of the orthogonal trajectories is of the form

$$F\left(x, y, -\frac{1}{y'}\right) = 0$$

The general integral of this equation

$$\Phi_1(x, y, C) = 0$$

gives the family of orthogonal trajectories.

Note : Following exact differentials must be remembered :

1. $x\,dy + y\,dx = d\,(xy)$
2. $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
3. $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
4. $\dfrac{x\,dy + y\,dx}{xy} = d\,(\ln xy)$
5. $\dfrac{dx + dy}{x + y} = d(\ln\,(x + y))$
6. $\dfrac{x\,dy - y\,dx}{xy} = d\left(\ln \dfrac{y}{x}\right)$
7. $\dfrac{y\,dx - x\,dy}{xy} = d\left(\ln \dfrac{x}{y}\right)$
8. $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1} \dfrac{y}{x}\right)$
9. $\dfrac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1} \dfrac{x}{y}\right)$
10. $\dfrac{x\,dx + y\,dy}{x^2 + y^2} = d(\ln \sqrt{x^2 + y^2})$
11. $d\left(-\dfrac{1}{xy}\right) = \dfrac{x\,dy + y\,dx}{x^2 y^2}$
12. $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dx - e^x dy}{y^2}$
13. $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^y dy - e^y dx}{x^2}$

EXERCISE - 1

Only One Correct Answer

1. Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop if the constant of proportionality is $K > 0$, is :

(a) $\frac{dr}{dt}+K=0$ (b) $\frac{dr}{dt}-K=0$ (c) $\frac{dr}{dt}=Kr$ (d) none of these

2. The general solution of the differential equation, $y' + y\,\phi'(x) - \phi(x).\,\phi'(x) = 0$ where $\phi(x)$ is a known function is :

(a) $y = ce^{-\phi(x)} + \phi(x) - 1$ (b) $y = ce^{+\phi(x)} + \phi(x) - 1$

(c) $y = ce^{-\phi(x)} - \phi(x) + 1$ (d) $y = ce^{-\phi(x)} + \phi(x) + 1$

where c is an arbitrary constant.

3. Orthogonal trajectories of family of the curve $x^{2/3} + y^{2/3} = a^{2/3}$, where '$a$' is any arbitrary constant, is :

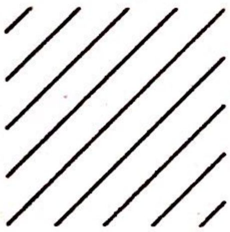
(a) $x^{2/3} - y^{2/3} = c$ (b) $x^{4/3} - y^{4/3} = c$

(c) $x^{4/3} + y^{4/3} = c$ (d) $x^{1/3} - y^{1/3} = c$

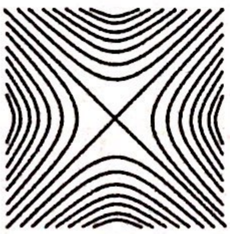
4. A function $y = f(x)$ satisfies the differential equation $f(x)\sin 2x - \cos x + (1+\sin^2 x)\,f'(x) = 0$ with initial condition $y(0) = 0$. The value of $f\left(\frac{\pi}{6}\right)$ is equal to :

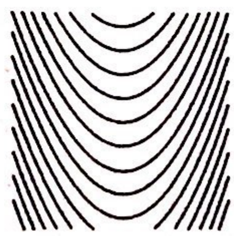
(a) $\frac{1}{5}$ (b) $\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $\frac{2}{5}$

5. The general solution of the differential equation $\frac{dy}{dx} = \frac{1-x}{y}$ is a family of curves which looks most like which of the following?

(a) 

(b) 

(c) 

(d) 

6. Given a curve C. Suppose that the tangent line at $P(x, y)$ on C is perpendicular to the line joining P and Q (1, 0). If the line $2x + 3y - 15 = 0$ is tangent to the curve C then the curve C denotes. :

(a) a circle touching the x-axis

(b) a circle touching the y-axis

(c) circle whose y-intercept is $4\sqrt{3}$

(d) a parabola with axis parallel to y-axis

7. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \frac{1}{15}$ then the time to drain the tank if the water is 4 meter deep to start with is :

(a) 30 min (b) 45 min (c) 60 min (d) 80 min

8. A function $y = f(x)$ satisfies the condition $f'(x) \sin x + f(x) \cos x = 1$, $f(x)$ being bounded when $x \to 0$. If $I = \int_0^{\pi/2} f(x)\, dx$ then :

(a) $\frac{\pi}{2} < I < \frac{\pi^2}{4}$ (b) $\frac{\pi}{4} < I < \frac{\pi^2}{2}$ (c) $1 < I < \frac{\pi}{2}$ (d) $0 < I < 1$

9. A curve satisfying the initial condition, $y(1) = 0$, satisfies the differential equation, $x\frac{dy}{dx} = y - x^2$. The area bounded by the curve and the x-axis is :

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

10. The equation of a curve passing through (1, 0) for which the product of the abscissa of a point P and the intercept made by a normal at P on the x-axis equals twice the square of the radius vector of the point P, is :

(a) $x^2 + y^2 = Cx^4$ (b) $x^2 + y^2 = 2x^4$ (c) $x^2 - y^2 = 4x^4$ (d) $x^2 - y^2 = x^4$

11. A wet porous substance in the open air loses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half its moisture during the first hour, then the time when it would have lost 99.9% of its moisture is : (weather conditions remaining same)

(a) more than 100 hours (b) more than 10 hours

(c) approximately 10 hours (d) approximately 9 hours

12. A curve C passes through origin and has the property that at each point (x, y) on it the normal line at that point passes through $(1, 0)$. The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is

(a) $x = 0$ (b) $y = 0$ (c) $y = x + 1$ (d) $x + y + 1 = 0$

13. Let g be a differentiable function satisfying $\int_0^x (x - t + 1)\, g(t)\, dt = x^4 + x^2$ for all $x \geq 0$. The value of $\int_0^1 \frac{12}{g'(x) + g(x) + 10}\, dx$ is equal to :

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

14. If $y = \frac{x}{\ln|cx|}$ (where c is an arbitrary constant) is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ then the function $\phi\left(\frac{x}{y}\right)$ is :

(a) $\frac{x^2}{y^2}$ (b) $-\frac{x^2}{y^2}$ (c) $\frac{y^2}{x^2}$ (d) $-\frac{y^2}{x^2}$

15. The curve, with the property that the projection of the ordinate on the normal is constant and has a length equal to 'a', is :

(a) $x + a\ln(\sqrt{y^2 - a^2} + y) = c$ (b) $x + \sqrt{a^2 - y^2} = c$

(c) $(y - a)^2 = cx$ (d) $ay = \tan^{-1}(x + c)$

16. The substitution $y = z^\alpha$ transforms the differential equation $(x^2y^2 - 1)dy + 2xy^3dx = 0$ into a homogeneous differential equation for :

(a) $\alpha = -1$ (b) 0 (c) $\alpha = 1$ (d) no value of α

17. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x t\, y(t)\, dt = x^2 y(x)$, $(x > 0)$ is :

(a) $x^2 + y^2 = 13$ (b) $y^2 = \frac{9}{2}x$

(c) $\frac{x^2}{8} + \frac{y^2}{18} = 1$ (d) $xy = C$

18. Solution of the differential equation $(e^{x^2} + e^{y^2})\, y \frac{dy}{dx} + e^{x^2}(xy^2 - x) = 0$, is :

(a) $e^{x^2}(y^2 - 1) + e^{y^2} = C$ (b) $e^{y^2}(x^2 - 1) + e^{x^2} = C$

(c) $e^{y^2}(y^2 - 1) + e^{x^2} = C$ (d) $e^{x^2}(y - 1) + e^{y^2} = C$

19. Let $\frac{x\,dy}{dx} - y = x^2(xe^x + e^x - 1)$ for all $x \in R - \{0\}$ such that $y(1) = e - 1$. If $y(2) = k\,y(1)$ $(y(1) + 2)$, then the value of k is :

(a) 1 (b) 2 (c) 3 (d) 4

20. Let a solution $y = y(x)$ of the differential equation, $dy + xy\,dx = x\,dx$ satisfy $y(0) = 2$, then the area enclosed by the curve $y = y(x)$, $x = 0$ and $y = 1$ in the first quadrant, is :

(a) $\int_0^\infty e^{\frac{-x^2}{2}}\,dx$ (b) $2\int_0^\infty e^{\frac{-x^2}{2}}\,dx$

(c) $\int_0^\infty \left(1 + e^{\frac{-x^2}{2}}\right)dx$ (d) $2\int_0^\infty \left(1 + e^{\frac{-x^2}{2}}\right)dx$

21. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k\,(T - t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then the scrap value $V(T)$ of the equipment is :

(a) $I - \frac{kT^2}{2}$ (b) $I - \frac{k(T-t)^2}{2}$

(c) e^{-kT} (d) $T^2 - \frac{I}{k}$

22. Let $y'(x) + \frac{g'(x)}{g(x)} \cdot y(x) = \frac{g'(x)}{1 + g^2(x)}$ where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function an R. If $g(1) = y(1) = 1$ and $g(e) = \sqrt{(2e - 1)}$ then $y(e)$ equals :

(a) $\frac{3}{2g(e)}$ (b) $\frac{1}{2g(e)}$ (c) $\frac{2}{3g(e)}$ (d) $\frac{1}{3g(e)}$

23. If $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$ and $y(0) = 1$, then :

(a) $y^2 = e^{2x} - 2e^{2x}\ln y$ (b) $y^2 = e^{2x} + 2e^{2x}\ln y$

(c) $y^2 = e^{2x} - \frac{1}{2}e^{2x}\ln y$ (d) $y^2 = e^{2x} + \frac{1}{2}e^{2x}\ln y$

24. A continuous function $f : R \to R$ satisfy the differential equation $f(x) = (1 + x^2)$ $\left[1 + \int_0^x \frac{f^2(t)}{1 + t^2}\,dt\right]$ then the value of $f(-2)$ is :

(a) 0 (b) $\frac{17}{15}$ (c) $\frac{-17}{15}$ (d) $\frac{15}{17}$

25. Let $f(x)\ (f(x) > 0)$ be a differentiable function satisfying

$$f^2(x) = \int_0^x (f^2(t) - f^4(t) + (f'(t))^2)\,dt + 100, \text{ where } f^2(0) = 100, \text{ then } \lim_{x\to\infty} f(x) \text{ can be :}$$

(a) 0 (b) 1 (c) $\frac{10}{9}$ (d) 10

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 to 3

Let $y = f(x)$ satisfies the equation $f(x) = (e^{-x} + e^{x})\cos x - 2x - \int_0^x (x-t)f'(t)\,dt$.

1. y satisfies the differential equation :

(a) $\frac{dy}{dx} + y = e^x(\cos x - \sin x) - e^{-x}(\cos x + \sin x)$

(b) $\frac{dy}{dx} - y = e^x(\cos x - \sin x) + e^{-x}(\cos x + \sin x)$

(c) $\frac{dy}{dx} + y = e^x(\cos x + \sin x) - e^{-x}(\cos x - \sin x)$

(d) $\frac{dy}{dx} - y = e^x(\cos x - \sin x) + e^{-x}(\cos x - \sin x)$

2. The value of $f'(0) + f''(0)$ equals :

(a) -1 (b) 2 (c) 1 (d) 0

3. $f(x)$ as a function of x equals :

(a) $e^{-x}(\cos x - \sin x) + \frac{e^x}{5}(3\cos x + \sin x) + \frac{2}{5}e^{-x}$

(b) $e^{-x}(\cos x + \sin x) + \frac{e^x}{5}(3\cos x - \sin x) - \frac{2}{5}e^{-x}$

(c) $e^{-x}(\cos x - \sin x) + \frac{e^x}{5}(3\cos x - \sin x) + \frac{2}{5}e^{-x}$

(d) $e^{-x}(\cos x + \sin x) + \frac{e^x}{5}(3\cos x - \sin x) - \frac{2}{5}e^{-x}$

Paragraph for Question Nos. 4 to 6

Let $g(x)$ be a polynomial of degree 3 passing through origin and have a local maximum at $x = \frac{-1}{2\sqrt{2}}$. Also $g'(x)$ has a local minimum at $x = 0$, and $g(1) = 5$.

4. The value of $g(2)$ is equal to :
   (a) 48 (b) 54 (c) 58 (d) 68

5. Let $f(x) = \text{sgn}(x)$ where sgn (x) denotes signum function of x, then which one of the following statement is **incorrect** for $f(g(x))$?
   (a) Area enclosed by $f(g(x))$ with x-axis between ordinates $x = -\alpha$ to $x = \alpha$ is 2α.
   (b) $\int_{-1}^{1} f(g(x))\, dx = 0$
   (c) $f(g(x))$ is a many-one function
   (d) $f(g(x))$ is periodic function

6. Consider a real-valued function $y = h(x)$ satisfying the differential equation $\frac{dy}{dx} + g'(x)\, y = (g(x) + 1)\; g'(x)$ such that $h(0) = 1$, then $h(1)$ equals :
   (a) $5 + e^{-5}$ (b) $11 + e^{11}$ (c) $-5 + e^{-5}$ (d) $11 + e^{-11}$

Paragraph for Question Nos. 7 to 9

For the curve $x^2y^3 = (2x + 3y)^5$, $\frac{dy}{dx} = \frac{-y}{g(x)}$ where $g(x)$ is a real-valued function.

Define $h(x) = 2g(x) + 3\,(g(x))^{\frac{2}{3}}$.

7. Which one of the following statement is correct for the function $h(x)$?
   (a) $x = -1$ is the point of maxima (b) $x = 1$ is the point of maxima
   (c) Non-derivable at $x = -1$ (d) $x = 0$ is point of maxima

8. Minimum value of $P(x) = \frac{(g(x))^2 + g(x) + 1}{x^2 + x + 1}$ is equal to :
   (a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) 5 (d) $\frac{1}{4}$

9. The ordinate of the point on the curve $y = h(x)$ where tangent is parallel to line $y = 2x + 4$, is :
   (a) $\frac{1}{2}$ (b) 4 (c) 3 (d) 1

Paragraph for Question Nos. 10 and 11

Let f be a function defined on the interval $[0, 2\pi]$ such that $\int_0^x (f'(t) - \sin 2t)\, dt = \int_x^0 f(t) \tan t\, dt$ and $f(0) = 1$.

10. The maximum value of $f(x)$ is :

(a) $\frac{5}{4}$ (b) $\frac{9}{8}$ (c) 1 (d) $\frac{1}{4}$

11. The number of solution of $f(x) = 1$ in interval $[0, 2\pi]$ is :

(a) 2 (b) 3 (c) 4 (d) 6

Paragraph for Question Nos. 12 to 14

Let $f : R^+ \to R$ be a differentiable function satisfying

$f(x) = e + (1 - x) \ln\left(\frac{x}{e}\right) + \int_1^x f(t)\, dt$ for all $x \in R^+$.

12. The value of definite integral $\int_0^1 f(x)\, dx$ is equal to :

(a) e (b) $e - 2$ (c) $e + 1$ (d) $\frac{e+1}{2}$

13. The x-intercept of normal drawn to the curve $y = f(x)$ at point P where $y = f(x)$ crosses the line $x = 1$, is equal to :

(a) $e^2 + e - 1$ (b) $\frac{e^2 + e + 1}{e + 1}$ (c) $\frac{e^2 - e + 1}{e + 1}$ (d) $e^2 + e + 1$

14. Which one of the following statement is correct for function $f(x)$?

(a) f has maxima but no minima (b) f has minima but no maxima

(c) f has neither maxima nor minima (d) f has both maxima and minima

Paragraph for Question Nos. 15 to 17

Let f be a differentiable function satisfying

$\int_0^{f(x)} f^{-1}(t)\, dt - \int_0^x (\cos t - f(t))\, dt = 0$ and $f(0) = 1$

15. The number of solution(s) of the equation $\begin{vmatrix} f(2x) & f(x) \\ \sin x & 2 \end{vmatrix} = 0$ in $(0, 2\pi)$ is :

(a) 2 (b) 3 (c) 4 (d) 5

16. The value of $\int_0^{\pi/2} f(x)\,dx$ lies in the interval :

(a) $\left(\frac{2}{\pi}, 1\right)$ (b) $\left(1, \frac{\pi}{2}\right)$ (c) $\left(\frac{3}{2}, \frac{\pi}{2}\right)$ (d) $\left(0, \frac{2}{\pi}\right)$

17. The value of $\lim_{x\to 0}\left(\left[\frac{\cos x}{f(x)}\right]+\left[\frac{\cos 2x}{f(2x)}\right]+\left[\frac{\cos 3x}{f(3x)}\right]+....+\left[\frac{\cos(100x)}{f(100x)}\right]\right)$ is equal to :

[**Note :** where $[k]$ denotes greatest integer less than or equal to k.]

(a) 0 (b) 4950 (c) 5049 (d) 5050

EXERCISE - 3

More Than One Correct Answers

1. A curve $y = f(x)$ has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of the point P from the x-axis. Then the differential equation of the curve :
   (a) is homogeneous
   (b) can be converted into linear differential equation with some suitable substitution
   (c) is the family of circles touching the x-axis at the origin
   (d) the family of circles touching the y-axis at the origin

2. The equation of the curve passing through (3, 4) and satisfying the differential equation, $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$ can be :
   (a) $x - y + 1 = 0$ (b) $x^2 + y^2 = 25$
   (c) $x^2 + y^2 - 5x - 10 = 0$ (d) $x + y - 7 = 0$

3. The area bounded by a curve, the axis of co-ordinates and the ordinate of some point of the curve is equal to the length of the corresponding arc of the curve. If the curve passes through the point $P(0, 1)$ then the equation of this curve can be :
   (a) $y = \frac{1}{2}(e^x - e^{-x} + 2)$ (b) $y = \frac{1}{2}(e^x + e^{-x})$
   (c) $y = 1$ (d) $y = \frac{2}{e^x + e^{-x}}$

4. A function $y = f(x)$ satisfying the differential equation $\frac{dy}{dx} \cdot \sin x - y\cos x + \frac{\sin^2 x}{x^2} = 0$ is such that, $y \to 0$ as $x \to \infty$ then the statement which is correct is :

(a) $\lim_{x\to 0} f(x) = 1$

(b) $\int_0^{\pi/2} f(x)\, dx$ is less than $\frac{\pi}{2}$

(c) $\int_0^{\pi/2} f(x)\, dx$ is greater than unity

(d) $f(x)$ is an odd function

5. Which of the following pair(s) is/are orthogonal?

(a) $16x^2 + y^2 = c$ and $y^{16} = kx$

(b) $y = x + ce^{-x}$ and $x + 2 = y + ke^{-y}$

(c) $y = cx^2$ and $x^2 + 2y^2 = k$

(d) $x^2 - y^2 = c$ and $xy = k$

where c and k arbitrary constant.

6. Let $\frac{dy}{dx} + y = f(x)$ where y is a continuous function of x with $y(0) = 1$ and $f(x) = \begin{cases} e^{-x} & \text{if } 0 \le x \le 2 \\ e^{-2} & \text{if } x > 2 \end{cases}$. Which of the following hold(s) good?

(a) $y(1) = 2e^{-1}$

(b) $y'(1) = -e^{-1}$

(c) $y(3) = -2e^{-3}$

(d) $y'(3) = -2e^{-3}$

7. Let C be the family of curves $f(x, y, c) = 0$ (no member of C is x-axis) such that length of subnormal at any point $P(x, y)$ on the curve C is equal to four times that of the length of subtangent at the same point. Which of the following statement(s) is(are) correct ?

(a) Equation of the line with positive y-intercept passing through (4, 2) and perpendicular to the curve C is $x + 2y = 8$

(b) Orthogonal trajectory of C is family of parallel lines having gradient ± 2

(c) Order and degree of the differential equation of family of curves C are 1 and 2 respectively

(d) Differential equation of family of curves is $2y' \pm x = 0$

8. Let function $y = f(x)$ satisfies the differential equation $x^2 \frac{dy}{dx} = y^2 e^{\frac{1}{x}}$ $(x \neq 0)$ and $\lim_{x\to 0^-} f(x) = 1$. Identify the correct statement(s) :

(a) Range of $f(x)$ is $(0, 1) - \left\{\frac{1}{2}\right\}$

(b) $f(x)$ is bounded

(c) $\lim_{x\to 0^+} f(x) = 1$

(d) $\int_0^{e} f(x)\, dx > \int_0^{1} f(x)\, dx$

9. If $2xy\,dy = (x^2 + y^2 + 1)\,dx$, $y(1) = 0$ and $y(x_0) = \sqrt{3}$, then x_0 can be :

(a) 2 (b) -2 (c) 3 (d) -3

10. If $y(x)$ satisfies the differential equation $\dfrac{dy}{dx} = \sin 2x + 3y \cot x$ and $y\left(\dfrac{\pi}{2}\right) = 2$, then which of the following statement(s) is (are) correct?

(a) $y\left(\dfrac{\pi}{6}\right) = 0$

(b) $y'\left(\dfrac{\pi}{3}\right) = \dfrac{9 - 3\sqrt{2}}{2}$

(c) $y(x)$ increases in interval $\left(\dfrac{\pi}{6}, \dfrac{\pi}{3}\right)$

(d) The value of definite integral $\displaystyle\int_{-\pi/2}^{\pi/2} y(x)\,dx$ equals π

EXERCISE - 4

Integer Answer Type

1. Let $y = f(x)$ be a curve C_1 passing through (2, 2) and $\left(8, \dfrac{1}{2}\right)$ and satisfying a differential equation $y\left(\dfrac{d^2y}{dx^2}\right) = 2\left(\dfrac{dy}{dx}\right)^2$. Curve C_2 is the director circle of the circle $x^2 + y^2 = 2$. If the shortest distance between the curves C_1 and C_2 is $(\sqrt{p} - q)$ where $p, q \in N$, then find the value of $(p^2 - q)$.
2. A function $y = f(x)$ satisfies $x f'(x) - 2f(x) = x^4 f^2(x)$, $\forall\ x > 0$ and $f(1) = -6$. Find the value of $f'(3^{1/5})$.
3. Let f be a continuous function satisfying the equation $\displaystyle\int_0^x f(t)\,dt + \int_0^x t\, f(x - t)\,dt = e^{-x} - 1$, then find the value of $e^{10} f(10)$.
4. Given a curve C. Let the tangent line at $P(x, y)$ on C is perpendicular to the line joining P and $Q(1, 0)$. If the line $2x + 3y - 15 = 0$ is tangent to the curve C then the length of the tangent from the point (5, 0) to the curve C is $\sqrt{n}$ (where $n \in N$). Find the value of n.
5. A normal is drawn at a point $P(x, y)$ on a curve. It meets the x-axis and the y-axis at A and B respectively such that $(x\text{-intercept})^{-1} + (y\text{-intercept})^{-1} = 1$, where O is origin, then find radius of the director circle of the curve passing through (3, 3).

6. Let $y = f(x)$ defined in [0, 2] satisfies the differential equation $y^3 y'' + 1 = 0$ where $f(x) \geq 0 \; \forall \, x \in D_f$ and $f'(1) = 0, f(1) = 1$ then find the maximum value of $f(x)$.
[**Note :** D_f denotes the domain of the function and y'' denotes the 2^{nd} derivative of y w.r.t. x.]

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (a)	**2.** (a)	**3.** (b)	**4.** (d)	**5.** (b)	**6.** (c)	**7.** (c)	**8.** (a)	**9.** (d)	**10.** (a)
11. (c)	**12.** (a)	**13.** (c)	**14.** (d)	**15.** (a)	**16.** (a)	**17.** (d)	**18.** (a)	**19.** (d)	**20.** (a)
21. (a)	**22.** (a)	**23.** (a)	**24.** (d)	**25.** (b)					

EXERCISE 2 : Linked Comprehension Type

1. (a)	**2.** (d)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (b)	**8.** (b)	**9.** (a)	**10.** (b)
11. (c)	**12.** (b)	**13.** (d)	**14.** (c)	**15.** (b)	**16.** (b)	**17.** (a)			

EXERCISE 3 : More Than One Correct Answers

1. (a, b, d)	**2.** (a, b)	**3.** (b, c)	**4.** (a, b, c)	**5.** (a, b, c, d)
6. (a, b, d)	**7.** (a, c)	**8.** (a, b, d)	**9.** (a, b)	**10.** (a, c)

EXERCISE 4 : Integer Answer Type

1. 62	**2.** 8	**3.** 9	**4.** 3	**5.** 4
6. 1				

□□□

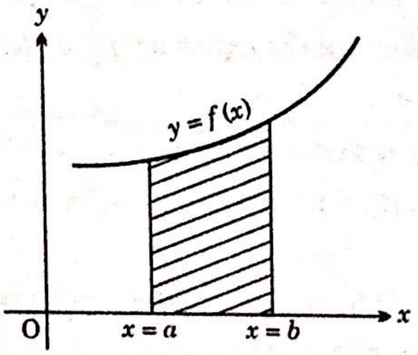
Chapter 9

Area Under Curve

KEY CONCEPTS

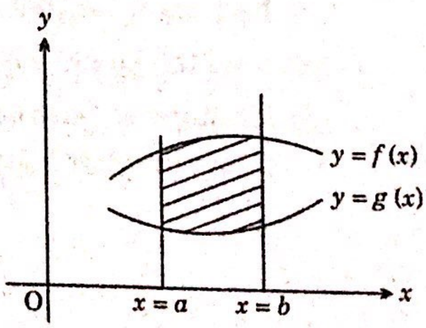
I. POINTS TO REMEMBER :

(i) The area bounded by the curve $y = f(x)$, the x-axis and the coordinates at $x = a$ and $x = b$ is given by, $A = \int_a^b f(x)\, dx = \int_a^b y\, dx.$



(ii) If the area is below the x-axis then A is negative. The convention is to consider the magnitude only *i.e.* $A = \left| \int_a^b y\, dx \right|$ in this case.

(iii) Area between the curves $y = f(x)$ and $y = g(x)$ between the coordinates at $x = a$ and $x = b$ is given by, $A = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx.$



(iv) Average value of a function $y = f(x)$ w.r.t. x over an interval $a \le x \le b$ is defined as:

$$y\,(av) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(v) The area function A_a^x satisfies the differential equation $\frac{dA_a^x}{dx} = f(x)$ with initial condition $A_a^a = 0$.

Note : If $F(x)$ is any integral of $f(x)$ then,

$$A_a^x = \int f(x)\,dx = F(x) + c$$

$$A_a^a = 0 = F(a) + c \Rightarrow c = -F(a)$$

hence $A_a^x = F(x) - F(a)$. Finally by taking $x = b$ we get, $A_a^b = F(b) - F(a)$.

2. CURVE TRACING :

The following outline procedure is to be applied in Sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

(iii) If powers of x and y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(v) If on interchanging the signs of x and y both the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(b) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x-axis and also the y-axis.

(d) Examine if possible the intervals when $f(x)$ is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

3. USEFUL RESULTS :

(a) Whole area of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

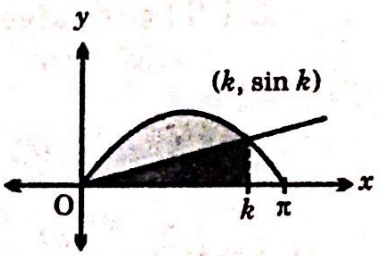
(b) Area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $16ab/3$.

(c) Area included between the parabola $y^2 = 4ax$ and the line $y = mx$ is $8a^2/3m^3$.

EXERCISE - 1

Only One Correct Answer

1. In the shown figure, half a period of $\sin x$ from 0 to π is split into two regions (light and dark shaded) of equal area by a line through the origin. If the line and the sine function intersect at a point whose x coordinate is k, then k satisfies the equation :



(a) $k\cos k + 2\sin k = 0$

(b) $k\sin k + 2\cos k = 0$

(c) $k\sin k + 2\cos k - 2 = 0$

(d) $2\cos k + k\sin k + 2 = 0$

2. If the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals $\frac{9}{2}$, then the value of m is :

(a) -4 (b) -2 (c) 2 (d) 5

3. Let A_n be the area of region bounded by a curve $y = x^3(1-x^2)^n$, $0 \le x \le 1$ and the x-axis, then the value of $\sum_{n=1}^{\infty} A_n$ is equal to :

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) 1

4. If the area enclosed by $g(x)$, $x = -3$, $x = 5$ and x-axis where $g(x)$ is the inverse of $f(x) = x^3 + 3x + 1$ is A, then $[A]$ equals :

[**Note:** $[k]$ denotes the greatest integer function less than or equal to k.]

(a) 2 (b) 3 (c) 4 (d) 5

5. The area enclosed by the curve $y^2 + x^4 = x^2$ is :

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$ (c) $\frac{8}{3}$ (d) $\frac{10}{3}$

6. Let Z be a complex number such that Re $(Z) = \sqrt{x^2+4}$ and Im $(Z) = \sqrt{y-4}$ satisfying $|Z| = \sqrt{10}$. Area enclosed by the set of points (x, y) on the complex plane, is :

(a) $8\sqrt{6}$ (b) $4\sqrt{6}$ (c) $\frac{20\sqrt{10}}{3}$ (d) $\frac{40\sqrt{10}}{3}$

7. Area enclosed by the curve $y = x^2 + 1$ and a normal drawn to it with gradient -1, is equal to :

(a) $\frac{8}{12}$ (b) $\frac{16}{12}$ (c) $\frac{19}{12}$ (d) $\frac{43}{12}$

8. Let 'a' be a positive constant number. Consider two curves $C_1 : y = e^x$, $C_2 : y = e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the y-axis, then $\lim_{a\to 0} \frac{S}{a^2}$ equals :

(a) 4 (b) $\frac{1}{2}$ (c) 0 (d) $\frac{1}{4}$

9. Let $f(x) = \int_0^x \frac{dt}{1+t^2}$. If the area of the figure surrounded by the normal line at $x = 1$ of $y = f(x)$, x-axis and the graph of $y = f(x)$ is $\frac{\pi^2}{k} + \frac{\pi}{4} - \ln\sqrt{2}$ sq. units (where $k \in N$) then the value of K is :

(a) 8 (b) 36 (c) 32 (d) 64

10. 3 points $O(0, 0), P(a, a^2), Q(-b, b^2)$ $(a > 0, b > 0)$ are on the parabola $y = x^2$. Let S_1 be the area bounded by the line PQ and the parabola and let S_2 be the area of the triangle OPQ, the minimum value of $\frac{S_1}{S_2}$ is :

(a) $\frac{4}{3}$ (b) $\frac{5}{3}$ (c) 2 (d) $\frac{7}{3}$

11. Area enclosed by the graph of the function $y = \ln^2 x - 1$ lying in the 4^{th} quadrant is :

(a) $\frac{2}{e}$ (b) $\frac{4}{e}$ (c) $2\left(e+\frac{1}{e}\right)$ (d) $4\left(e-\frac{1}{e}\right)$

12. The area bounded by $y = 2 - |2-x|$ and $y = \frac{3}{|x|}$ is :

(a) $\frac{4+3\ln 3}{2}$ (b) $\frac{4-3\ln 3}{2}$ (c) $\frac{3}{2}+\ln 3$ (d) $\frac{1}{2}+\ln 3$

13. The area of the region of the plane bounded by $(|x|, |y|) \le 1$ and $xy \le \frac{1}{2}$ is :

(a) $4 - \ln 2$ (b) $\frac{15}{4}$ (c) $2 + 2\ln 2$ (d) $3 + \ln 2$

14. Let $f(x) = x^2 + 6x + 1$ and R denote the set of points (x, y) in the coordinate plane such that $f(x) + f(y) \le 0$ and $f(x) - f(y) \le 0$. The area of R is equal to :

(a) 16π (b) 12π (c) 8π (d) 4π

15. The graphs of $f(x) = x^2$ and $g(x) = cx^3$ $(c > 0)$ intersect at the points $(0, 0)$ and $\left(\frac{1}{c}, \frac{1}{c^2}\right)$. If the region which lies between these graphs and over the interval $\left(0, \frac{1}{c}\right)$ has the area equal to $\frac{2}{3}$ then the value of c is :

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 2

16. The positive values of the parameter 'a' for which the area of the figure bounded by the curve $y = \cos ax$, $y = 0$, $x = \frac{\pi}{6a}$, $x = \frac{5\pi}{6a}$ is greater than 3 are :

(a) ϕ (b) $\left(0, \frac{1}{3}\right)$ (c) $(3, \infty)$ (d) none of these

17. $y = f(x)$ is a function which satisfies :

(i) $f(0) = 0$

(ii) $f''(x) = f'(x)$ and

(iii) $f'(0) = 1$

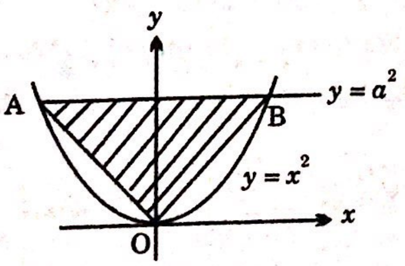
then the area bounded by the graph of $y = f(x)$, the lines $x = 0$, $x - 1 = 0$ and $y + 1 = 0$, is :

(a) e (b) $e - 2$ (c) $e - 1$ (d) $e + 1$

18. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$ is :

(a) 1 (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) 2

19. In the given figure, if A_1 is the area of the ΔAOB and A_2 is the area of the parabolic region AOB then the ratio $\frac{A_1}{A_2}$ as $a \to 0$ is :



(a) 1 (b) $\frac{8}{9}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$

20. If the tangent to the curve $y = 1 - x^2$ at $x = \alpha$, where $0 < \alpha < 1$, meets the axes at P and Q. Also α varies, the minimum value of the area of the triangle OPQ is k times the area bounded by the axes and the part of the curve for which $0 < x < 1$, then k is equal to :

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{75}{16}$ (c) $\frac{25}{18}$ (d) $\frac{2}{3}$

21. Consider the following regions in the plane :

$R_1 = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$

$R_2 = \{(x, y) : x^2 + y^2 \le 4/3\}$

The area of the region $R_1 \cap R_2$ can be expressed as $\frac{a\sqrt{3} + b\pi}{9}$, where a and b are integers. Then the value of $(a + b)$ equals :

(a) 2 (b) 3 (c) 4 (d) 5

22. Infinite rectangles each of width 1 unit and height $\left(\frac{1}{n} - \frac{1}{n+1}\right)$ $(n \in N)$ are constructed such that ends of exactly one diagonal of every rectangle lies along the curve $y = \frac{1}{x}$. The sum of areas of all such rectangles, is :

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) 1

23. The area bounded by the x-axis and the part of graph of $y = \cos x$ between $x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$ is separated into two regions by the line $x = k$. If the area of the region for $\frac{-\pi}{2} \le x \le k$ is three times the area of the region for $k \le x \le \frac{\pi}{2}$, then k is equal to :

(a) $\arcsin\left(\frac{1}{4}\right)$ (b) $\arcsin\left(\frac{1}{3}\right)$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

24. Let $d_1[(x_1, y_1), (x_2, y_2)] = |x_1 - x_2| + |y_1 - y_2|$ and

$d_2[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

denote the distance between (x_1, y_1) and (x_2, y_2) on the coordinate plane. The area of the region enclosed by the set of points (x, y) satisfying $d_1[(x, y), (0, 0)] \ge 1$ and $d_2[(x, y), (0, 0)] \le 1$, is :

(a) $\pi - 2$ (b) $\pi + 2$ (c) $\pi + 4$ (d) $\pi - 4$

25. The slope of the tangent to the curve $y = f(x)$ at $[x, f(x)]$ is $(2x + 1)$. If the curve passes through the point (1, 3) then the area bounded by the curve $y = f(x)$ and the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$, is equal to :

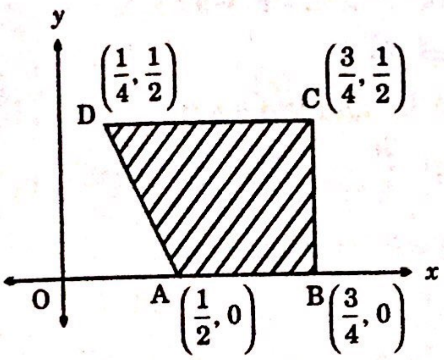
(a) $\frac{5}{6}$ (b) $\frac{6}{5}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

26. Let $2a > -1$. If the area of the region of the plane defined by $\{(x, y) : x \geq 0, 2y - x \geq 0, ax + y - 3 \leq 0\}$ is equal to 3, then the value of a, lies in :

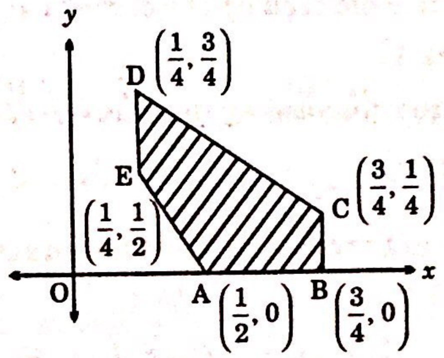
(a) $\left(\frac{1}{2}, \frac{3}{4}\right)$ (b) $\left(\frac{3}{4}, \frac{3}{2}\right)$ (c) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (d) $\left(\frac{5}{2}, \frac{10}{3}\right)$

27. Which of the following shaded region in xy plane represents points satisfying $x + y \leq 1$, $\ln(2x + y) \geq 0$ and $16x^2 - 16x + 3 \leq 0$?

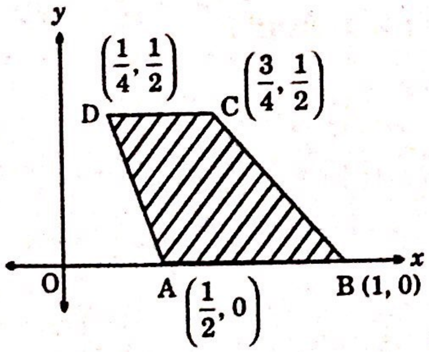
(a)



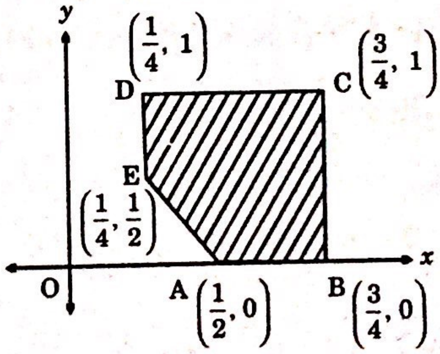
(b)



(c)



(d)



28. If $S(a)$ is the algebraic area bounded by the curve $y = e^{x-a}$, $y = \frac{x}{a}$ and the y-axis where $a > 0$, then $\lim_{a \to \infty} \frac{S(a)}{a}$ is equal to :

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{-1}{2}$ (d) does not exist

EXERCISE - 2

Linked Comprehension Type

Paragraph for Question Nos. 1 to 3

Let $y = f(x)$ be a solution of the differential equation $\frac{dy}{dx} = -4x(y-2)$ with $f(0) = 2$ and $g(x) = \max.\left\{1+\frac{x}{2}-\left[\frac{x}{2}\right], \sin x, |x-1|\right\}$, $h(x) = \ln x$, $k(x) = \min.\{x^2 - 3x + 2, 2 + \sin x\}$

[Note : [y] denotes greatest integer function less than or equal to y.]

1. Area enclosed by the curves $y = f(x)$, $y = g(x)$ and the y-axis, is :

   (a) 1 (b) 2 (c) 3 (d) 4

2. Area bounded by the curve $y = f(x)$, $y = h(x)$ and the y-axis, is :

   (a) e (b) e^2 (c) $\frac{3e}{2}$ (d) $e^2 - 1$

3. Number of solutions of the equation $f(x) = k(x)$ in $[0, 2\pi]$, is :

   (a) 2 (b) 3 (c) 4 (d) 5

Paragraph for Question Nos. 4 and 5

Let $f(x)$ $(x \geq 1)$ be a differentiable function satisfying

$$f(x) = (\log_e x)^2 - \int_1^e \frac{f(t)}{t}\,dt.$$

4. $\lim_{x\to e}\left[f(x)+\frac{1}{6}\right]^{\frac{1}{x-e}}$ is equal to :

   (a) $\frac{2}{e}$ (b) $\frac{1}{e}$ (c) $e^{\frac{1}{e}}$ (d) $e^{\frac{2}{e}}$

5. Area bounded by tangent line of $y = f(x)$ at the point $[e, f(e)]$, the curve $y = f(x)$ and the line $x = 1$, is :

   (a) $e+\frac{1}{e}$ (b) $e+\frac{1}{e}-1$

   (c) $e+\frac{1}{e}-2$ (d) $e+\frac{1}{e}-3$

Paragraph for Question Nos. 6 to 8

Let $y'(x)\,\phi(x) - y(x)\,\phi'(x) + y^2(x) = 0$, $y(1) = 1$, $x \in R$, where $f'(x)$ denotes $\dfrac{df(x)}{d(x)}$ and $\phi(x)$ is a given non-constant differentiable function on R with $\phi(1) = 0$ and $\phi(2) = 4$.

6. The value of $y(2)$ is equal to :

(a) 0 (b) 1 (c) 2 (d) 4

7. The value of definite integral $\displaystyle\int_1^2 \left(\frac{\phi(x)}{y(x)} \cdot \frac{1}{x^2\sqrt{x^2 + \left(\frac{\phi(x)}{y(x)}\right)^2}}\right) dx$, is equal to :

(a) $\sqrt{5}$ (b) $\dfrac{\sqrt{5}}{2} - 1$ (c) $\sqrt{5} - 1$ (d) $\sqrt{5} - 2$

8. Which one of the following statement is correct for the function $g(x) = \dfrac{\phi(x)}{y(x)}$?

(a) $g(x)$ is decreasing on R

(b) Area bounded by $g(x)$ and coordinate axes is $\dfrac{1}{4}$

(c) $g(x)$ has minima but no maxima

(d) $|g(x)|$ is continuous but not differentiable on R

Paragraph for Question Nos. 9 to 11

Consider three real-valued functions f, g and h defined on R (the set of real numbers). Let $f(x) = 2x^3 + 3\left(1 - \dfrac{3a}{2}\right)x^2 + 3(a^2 - a)x + b$ where $a, b \in R$ and $g(x) = \dfrac{f'(x)}{6}$. Also $h(x)$ is such that $h''(x) = 6x - 4$ and $h(x)$ has a local minimum value 5 at $x = 1$.

9. The true set of values of a for which $f(x)$ has negative point of local minimum, is :

(a) $(-\infty, 0)$ (b) $(1, \infty)$ (c) $(0, 1)$ (d) $(1, \infty) - \{2\}$

10. The complete set of values of a for which vertex of parabola $y = g(x)$ has negative coordinate, is :

(a) $\left(\dfrac{1}{2}, 1\right)$ (b) $(0, \infty)$ (c) $R - \{2\}$ (d) $(1, \infty) - \{2\}$

11. The area bounded by $y = h(x)$ between $x = 0$ and $x = 2$, is :

(a) $\frac{23}{3}$ (b) $\frac{20}{3}$ (c) $\frac{40}{3}$ (d) $\frac{32}{3}$

Paragraph for Question Nos. 12 to 14

Let $P(x)$ be a biquadratic function of x such that $\lim_{x\to 0}\left[\frac{P(-x)}{2x^3}\right]^{\frac{1}{x}} = \frac{1}{e^3}$.

12. The value of $P(1)$ is equal to :

(a) 0 (b) 8 (c) –8 (d) –6

13. Which one of the following statement is correct for $P(x)$?

(a) $P(x)$ has minima but no maxima

(b) $P(x)$ has maxima but no minima

(c) $P(x)$ increases in $\left(-\infty, \frac{1}{4}\right)$ and decreases in $\left(\frac{1}{4}, \infty\right)$

(d) $P(x)$ decreases in $\left(-\infty, \frac{-1}{4}\right)$ and increases in $\left(\frac{-1}{4}, \infty\right)$

14. The area bounded by $y = P(x)$ and x-axis in the second quadrant is equal to :

(a) $\frac{1}{810}$ (b) $\frac{1}{243}$ (c) $\frac{1}{270}$ (d) $\frac{1}{81}$

Paragraph for Question Nos. 15 to 18

Let $f(x)$ be a differentiable function such that $f(x+y) = e^x f(y) + e^y f(x)$ all x, y and $f'(0) = 1$.

15. $f(x)$ has :

(a) maximum (b) minimum

(c) both maximum and minimum (d) neither maximum nor minimum

16. The range of $f(x)$ is :

(a) R (b) $[0, \infty)$ (c) $\left(-\frac{1}{e}, 1\right)$ (d) $\left[-\frac{1}{e}, \infty\right)$

17. $\lim_{x\to -\infty} f(x)$ is :

(a) 0 (b) 1 (c) –1 (d) non existent

18. The area bounded by the curve $y = f(x)$ and the x-axis is :

(a) 1 (b) $\frac{1}{2}$ (c) 2 (d) e

EXERCISE - 3

More Than One Correct Answers

1. Let T be the triangle with vertices $(0, 0)$, $(0, c^2)$ and (c, c^2) and let R be the region between $y = cx$ and $y = x^2$ where $c > 0$ then :

(a) Area $(R) = \frac{c^3}{6}$

(b) Area of $R = \frac{c^3}{3}$

(c) $\lim_{c \to 0^+} \frac{\text{Area } (T)}{\text{Area } (R)} = 3$

(d) $\lim_{c \to 0^+} \frac{\text{Area } (T)}{\text{Area } (R)} = \frac{3}{2}$

2. Area of the region bounded by the curve $y = \tan x$ and lines $y = 0$ and $x = 1$ is equal to :

(a) $\int_0^1 \tan(1 - x)\, dx$

(b) $\int_0^{\tan 1} \tan^{-1} y\, dy$

(c) $\int_0^1 \tan^{-1} x\, dx$

(d) $\tan 1 - \int_0^{\tan 1} \tan^{-1} x\, dx$

3. Let $f(x)$ be a differentiable function satisfying $\int_0^{2x} x f(t)\, dt + 2\int_x^0 t f(2t)\, dt$ $= 2x^4 - 2x^3$, for all $x \in R$ then which of the following is/are correct?

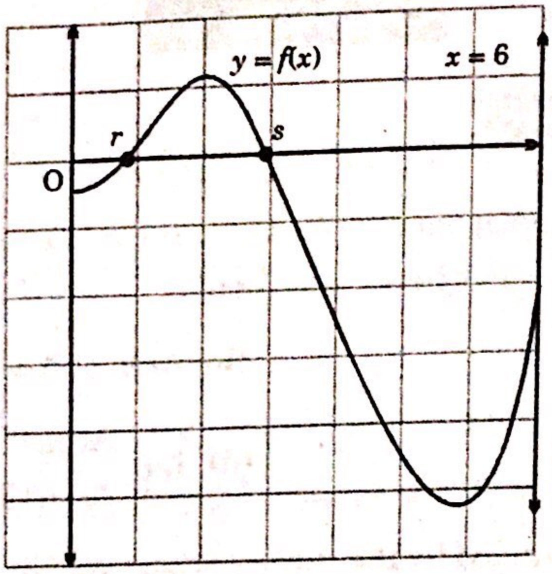
(a) Minimum value of $f(x)$ is equal to $\frac{-3}{4}$

(b) $f(|x|)$ is non-derivable at exactly one value of x

(c) Area bounded by $y = f(x)$ and x-axis is equal to 2

(d) $\lim_{x \to 1} \frac{f(x)}{x - 1}$ exist and is equal to 3

4. The graph of $y = f(x)$ is shown with roots r and s $(r < s)$. Area bounded by the graph of $f(x)$, x-axis, $x = 0$ and $x = 6$ over the intervals $[0, r]$ and $[r, s]$ and $[s, 6]$ are $\frac{2}{5}$, 2 and 12 respectively. If $m = \int_s^0 f(x)\, dx$, $p = \int_r^6 f(x)\, dx$, $q = \left|\int_0^6 f(x)\, dx\right|$, $n = \int_0^6 |f(x)|\, dx$, then :



(a) $p < m$ (b) $m < 1$ (c) $q < 11$ (d) $n > 11$

5. The function $y = f(x)$ is continuous over [0, 10]. If the area bounded by $f(x)$ and the x-axis between $x = 0$ and $x = 10$ is S where $-10 \le f(x) \le 10$, then which of the following is/are not always correct?

(a) $S \le 100$ (b) $\left|\int_0^{10} f(x)\,dx\right| = S$ (c) $\int_0^{10} f(x)\,dx = S$ (d) $S \le \int_0^{10} f(x)\,dx$

EXERCISE - 4

Integer Answer Type

1. The parabola $P: y = ax^2$ where 'a' is a positive real constant, is touched by the line $L: y = mx - b$ (where m is a positive constant and b is real) at the point T. Let Q be the point of intersection of the line L and the y-axis is such that $TQ = 1$. If A denotes the maximum value of the region surrounded by P, L and the y-axis, find the value of $\frac{1}{A}$.

2. If the area enclosed by the curve $y^2 = 4x$ and $16y^2 = 5(x-1)^3$ can be expressed in the form $\frac{L\sqrt{M}}{N}$ where L and N are relatively prime and M is a prime, find the value of $(L + M + N)$.

3. Consider the collection of all curves of the form $y = a - bx^2$ that pass through the point (2, 1) where a and b are positive real numbers. If the minimum area of the region bounded by $y = a - bx^2$ and the x-axis is $\sqrt{A}$, find the value of $A \in N$.

4. Let $f(x) = x - x^2$ and $g(x) = ax$. If the area bounded by $f(x)$ and $g(x)$ is equal to the area bounded by the curves $x = 3y - y^2$ and $x + y = 3$, then find the value of $|[a]|$:
[**Note :** $[k]$ denotes the greatest integer less than or equal to k.]

5. Let C_1 and C_2 be two curves which satisfy the differential equation $\left|x - y\frac{dx}{dy}\right| = 2\left|\frac{dy}{dx}\right|$ and passes through M (1, 1). If the area enclosed by curves C_1, C_2 and co-ordinate axes is $\frac{m}{n}$ $(m, n \in N)$ then find the least value of $(m + n)$.

6. If the area bounded by the curve $y = |\cos^{-1}(\sin x)| - |\sin^{-1}(\cos x)|$ and x-axis from $\frac{3\pi}{2} \le x \le 2\pi$, is equal to $\frac{\pi^2}{k}$, where $k \in N$, then find k.

7. Let A_n be the area bounded by the curve $y = x^n$ $(n \ge 1)$ and the line $x = 0, y = 0$ and $x = \frac{1}{2}$. If $\sum_{n=1}^{n} \frac{2^n A_n}{n} = \frac{1}{3}$ then find the value of n.

8. Let $y = f(x) = \begin{cases} \sqrt{x+3}, & -3 \le x < -2 \\ -1 + \sqrt{x+2}, & -2 \le x < -1 \\ -2 + \sqrt{x+1}, & -1 \le x \le 0 \end{cases}$. If $|y| = f(-|x|)$ be a curve and area enclosed between the curve and the circle $x^2 + y^2 = 5$ equals $p + \pi q$, where p and q are integers then find the value of $(p + q)$.

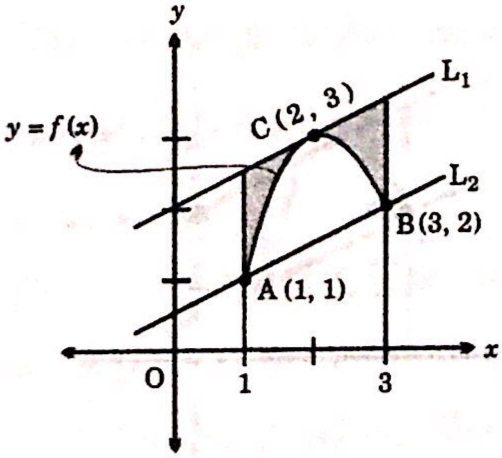
9. Let $f : R^+ \to R$ be a differentiable function satisfying $f(x) = e + (1 - x) \ln\left(\frac{x}{e}\right) + \int_1^x f(t)\,dt \ \forall \ x \in R^+$. If the area enclosed by the curve $g(x) = x\,[f(x) - e^x]$ lying in the fourth quadrant is A, then find the value of A^{-2}.

10. Let $f(x)$ be a polynomial of degree 3. If the curve $y = f(x)$ has relative extrema at $x = \frac{\pm 2}{\sqrt{3}}$ and passes through (0, 0) and (1, – 2) dividing the circle $x^2 + y^2 = 4$ in two parts, then the area bounded by $x^2 + y^2 = 4$ and $y \ge f(x)$ is $\frac{k\pi}{2}$. Find the value of k.

11. If the maximum area of the region enclosed by the curves $y = |x|\, e^{|x|}$ and the line $y = a$ $(0 \le a \le e)$ in $x \in [-1, 1]$ is A, then find the value of $[A]$.
[**Note :** $[k]$ denotes greatest integer function less than or equal to k.]

12. Let $y = f(x)$ be a real-valued differentiable function on R (the set of all real numbers) such that $f(1) = 1$. If $f(x)$ satisfies $x f'(x) = x^2 + f(x) - 2$ then find the area enclosed by $f(x)$ with x-axis between coordinates $x = 0$ and $x = 3$.

13. The following figure shows the graph of a continuous function $y = f(x)$ on the interval $[1, 3]$. The points A, B, C have co-ordinates $(1, 1), (3, 2), (2, 3)$ respectively and the lines L_1 and L_2 are parallel with L_1 being tangent to the curve at C. If the area under the graph of $y = f(x)$ from $x = 1$ to $x = 3$ is 4 square units, then find the area (in square units) of shaded region.



14. Let $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right), g(x) = \cot^{-1}\left(\frac{2x}{x^2-1}\right)$ where $x \in (-1, 1)$. If area bounded by the curves $y = f(x) + g(x)$ and $y = \pi x^2$ is A then find the value of $[A]$.

[**Note :** $[K]$ denotes greatest integer less than or equal to K.]

15. Let f_1, f_2 and f_3 be three curves satisfying the differential equation $y(1-y^2)\,dx = x(y^2+1)\,dy$. If f_3 cuts the curves f_1 and f_2 at A and B respectively and one of the curves is passing through $C(2, -1)$, then find the area of ΔABC.

16. Let $P(x)$ be a polynomial function of degree 'n' satisfying $[P(x)]^2 . P'''(x) = [P''(x)]^3 P'(x) \ \forall \ x \in R$. Let $f(x)$ a polynomial whose degree is same as of $P(x)$. If the area bounded by $y = f(x)$, the x-axis and the coordinates of two local minima is $\frac{p}{q}$ where p and q are co-prime then find the value of $(p - q)$. Given $f'(0) = f'(-1) = f'(1) = 0$, $f(0) = 4$ and $f(1) = f(-1) = 3$.

17. If the area bounded by the curves $f(x) = [\cos^{-1}|\cos x|]^2, g(x) = [\cos^{-1}|\cos x|]$ and $|x| = \frac{\pi}{2}$ is $a\pi^3 + b\pi^2 + c$, then find the minimum value of $(|a|+|b|+|c|)$.

18. If $\int_0^1 [4x^3 - f(x)]\, f(x)\, dx = \frac{4}{7}$ then find the area of region bounded by $y = f(x)$, x-axis and coordinate $x = 1$ and $x = 2$.

ANSWERS

EXERCISE 1 : Only One Correct Answer

1. (b)	2. (b)	3. (c)	4. (c)	5. (b)	6. (a)	7. (b)	8. (d)	9. (d)	10. (a)
11. (b)	12. (b)	13. (d)	14. (c)	15. (c)	16. (b)	17. (c)	18. (d)	19. (c)	20. (a)
21. (c)	22. (d)	23. (c)	24. (a)	25. (d)	26. (b)	27. (b)	28. (c)		

EXERCISE 2 : Linked Comprehension Type

1. (a)	2. (b)	3. (c)	4. (d)	5. (d)	6. (d)	7. (b)	8. (d)	9. (a)	10. (c)
11. (d)	12. (c)	13. (b)	14. (a)	15. (b)	16. (d)	17. (a)	18. (a)		

EXERCISE 3 : More Than One Correct Answers

1. (a, c)	2. (a, b)	3. (a, b, d)	4. (a, b, c, d)	5. (b, c, d)

EXERCISE 4 : Integer Answer Type

1. 0012	2. 0124	3. 0048	4. 0001	5. 0009
6. 0004	7. 0002	8. 0015	9. 0016	10. 0004
11. 0003	12. 0006	13. 0002	14. 0004	15. 002
16. 0091	17. 0001	18. 15/2		

❑❑❑

PRACTICE TEST PAPER - 1

FOR JEE (MAIN)

INSTRUCTIONS :

(i) The question paper consists of **'30'** objective type questions.

(ii) Each question has four choices (a), (b), (c) and (d) out of which **Only One** is correct.

(iii) You will be awarded **4 marks** for each question, if you have darkened only the bubble corresponding to the correct answer and zero mark if no bubble are darkened. In all other cases, **minus one (–1) mark** will be awarded.

(iv) There is only one correct response for each question. Filling up more than one response in each question will be treated as wrong response and marks for wrong response will be deducted accordingly as per instruction (iii) above.

1. Let $f: R \to R$ be a periodic function such that

$f(T+x) = 1 + \{1 - 3f(x) + 3f(x)^2 - f(x)^3\}^{1/3}$ where T is a fixed positive number, then period of $f(x)$ is :

(a) T (b) $2T$ (c) $3T$ (d) $4T$

2. If z_1 and z_2 $(z_1, z_2 \neq 0)$ are two complex numbers such that $\left|\dfrac{z_1 - z_2}{z_1 + z_2}\right| = 1$ then :

(a) $z_2 = ikz_1 \ (k \in R)$ (b) $z_2 = kz_1 \ (k \in R)$

(c) $z_2 = z_1$ (d) $\dfrac{z_1}{z_2}$ is real

3. If $ax^2 + bx + 6 = 0$ does not have two distinct real roots, $a \in R, b \in R$ then the least value of $(3a + b)$ is :

(a) 4 (b) –1 (c) 1 (d) –2

4. An aeroplane flying with uniform speed horizontally 1 km above the ground is observed at an elevation of 60°. After 10 sec, if the elevation is observed to be 30°, then the speed of the plane (in km/hr) is :

(a) $\dfrac{240}{\sqrt{3}}$ (b) $200\sqrt{3}$ (c) $240\sqrt{3}$ (d) $\dfrac{120}{\sqrt{3}}$

5. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real less than 3 then largest integral value of 'a' is :

(a) -1 (b) 0 (c) 1 (d) 2

6. A skew - symmetric matrix A satisfies the relation $A^2 + I = 0$, where I is a unit matrix. Then A is :

(a) Idempotent matrix (b) Orthogonal matrix

(c) Nilpotent matrix (d) Periodic matrix

7. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ then $a^2 + b^2 + c^2 + 2abc$ is equal to:

(a) 2 (b) 1 (c) 0 (d) -1

8. Suman writes letters to his five friends. The number of ways can be letters be placed in the envelopes so that at least two of them are in the wrong envelopes are :

(a) 119 (b) 120 (c) 125 (d) 130

9. The sum of $\sum_{r=0}^{n}(-1)^r \frac{{}^nC_r}{{}^{r+2}C_r}$ is equal to :

(a) $\frac{2}{n+1}$ (b) $\frac{2}{n-1}$ (c) $\frac{2}{n+2}$ (d) $\frac{2}{n-2}$

10. The sum of first two terms of a geometric progression is 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative then the first term is :

(a) -4 (b) -12 (c) 4 (d) 12

11. $\lim_{n \to \infty} \sum_{r=1}^{n} \cot^{-1}\left(r^2 + \frac{3}{4}\right)$ is equal to :

(a) 0 (b) $\frac{\pi}{4}$ (c) $\tan^{-1} 2$ (d) $\tan^{-1} 3$

12. Let f be a differentiable function such that $f(x + y) = f(x) + f(y) + 2xy - 1$, for all real x and y. If $f'(0) = \cos\alpha$, then $\forall x \in R$:

(a) $f(x) < 0$ (b) $f(x) = 0$ (c) $f(x) > 0$ (d) $f(x) = x$

13. Let $f(x) = x^2 - 8x + 12, x \in [2,6]$.

Statement-1: $f'(c) = 0$, for some $c \in (2,6)$.

Statement-2: f is continuous on $[2, 6]$ and differentiable on $(2, 6)$ with $f(2) = f(6)$.

(a) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1

(b) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1

(c) Statement-1 is true, statement-2 is false

(d) Statement-1 is false, statement-2 is true

14. Let $h(x) = xg(x)$ where g is the inverse of $f(x)$. Also the values of $f(x)$ and $f'(x)$ are given as

x	2	3	5
$f(x)$	4	5	1
$f'(x)$	–1	2	3

then the value of $h'(5)$ equals :

(a) $\frac{11}{2}$ (b) $\frac{9}{2}$ (c) 5 (d) $\frac{5}{2}$

15. Let f be a function which is continuous and differentiable for all real x. If $f(2) = -4$ and $f'(x) \geq 6$ for all $x \in [2,4]$, then :

(a) $f(4) < 8$ (b) $f(4) \geq 8$ (c) $f(4) \geq 12$ (d) $f(4) < 12$

16. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real roots of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$:

(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of $P(x)$

(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of $P(x)$

(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of $P(x)$

(d) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of $P(x)$

17. The range of parameter 'b' for which the function $f(x) = \int_0^x (bt^2 + b + \cos t)dt$ is monotonic for all real values of x, is :

(a) $[-1,1]$ (b) $(-\infty,-1] \cup [1,\infty)$ (c) $(-\infty,-1) \cup (1,\infty)$ (d) $(-1,1)$

18. $\int x\{f(x^2) \cdot g''(x^2) - f''(x^2) \cdot g(x^2)\}dx$ is equal to :

(a) $f(x^2) \cdot g'(x^2) - g(x^2)f'(x^2) + C$ (b) $\frac{\left(f(x^2)g(x^2)f'(x^2)\right)}{2} + C$

(c) $\frac{\left(f(x^2)g'(x^2) - f'(x^2)g(x^2)\right)}{2} + C$ (d) $f(x^2) \cdot g(x^2)f'(x^2) + C$

19. If $\int_0^{100} f(x)dx = a$, then $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)dx\right)$ is equal to :

(a) 0 (b) a (c) $\frac{a}{2}$ (d) $2a$

20. If $u_n = \int_0^{\frac{\pi}{2}} x^n \sin x dx$, then the value of $(u_{10} + 90u_8)$ is :

(a) $9\left(\frac{\pi}{2}\right)^9$ (b) $10\left(\frac{\pi}{2}\right)^9$ (c) $\left(\frac{\pi}{2}\right)^9$ (d) $9\left(\frac{\pi}{2}\right)^8$

21. The area bounded by the circle $x^2 + y^2 = 8$, the parabola $x^2 = 2y$ and the line $y = x$ in $y \geq 0$ is :

(a) $\frac{2}{3} + 2\pi$ (b) $\frac{2}{3} - 2\pi$ (c) $\frac{2}{3} + \pi$ (d) $\frac{2}{3} - \pi$

22. The equation of the curve passing through (2, 5) and having the area of triangle formed by the x-axis the ordinate of a point on the curve and the tangent at the point 5 square units is :

(a) $xy = 10$ (b) $x^2 = 10y$ (c) $y^2 = 10x$ (d) $x^2y = 10$

23. $A(0,0)$, $B(2,1)$ and $C(3,0)$ are the vertices of a ΔABC and BD is its altitude. If the line through D parallel to the side AB intersects the side BC at a point K, then the product of the areas of the triangle ABC and BDK is :

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{1}{4}$

24. The common chord of the circle $x^2 + y^2 + 8x + 4y - 5 = 0$ and a circle passing through the origin and touching the line $y = x$, passes through the fixed point :

(a) $\left(\frac{5}{12}, \frac{5}{12}\right)$ (b) $\left(\frac{5}{12}, \frac{-5}{12}\right)$ (c) $\left(\frac{-5}{15}, \frac{5}{12}\right)$ (d) $\left(\frac{-5}{12}, \frac{-5}{12}\right)$

25. An ellipse has eccentricity $\frac{1}{2}$ and focus at the point $P\left(\frac{1}{2}, 1\right)$ its one directrix is the common tangent at the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse in standard form is :

(a) $9\left(x - \frac{1}{3}\right)^2 + (y-1)^2 = 1$ (b) $9\left(x - \frac{1}{3}\right)^2 + 12(y-1)^2 = 1$

(c) $\frac{\left(x - \frac{1}{3}\right)^2}{4} + \frac{(y-1)^2}{3} = 1$ (d) $\left(x - \frac{1}{3}\right)^2 + 9(y-1)^2 = 1$

26. If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other then a vector $\vec{v}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying the equations $\vec{v}\cdot\vec{a}=0, \vec{v}\cdot\vec{b}=1$ and $\left[\vec{v}\ \vec{a}\ \vec{b}\right]=1$ is :

(a) $\frac{1}{|b|^2}\vec{b}+\frac{1}{\left|\vec{a}\times\vec{b}\right|^2}\left(\vec{a}\times\vec{b}\right)$ (b) $\frac{\vec{b}}{|b|}+\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|^2}$

(c) $\frac{\vec{b}}{|b|^2}+\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$ (d) $\frac{\vec{b}}{|b|}+\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$

27. The planes $3x-y+z+1=0, 5x+y+3z=0$ intersect in the line PQ. The equation of the plane through the point (2, 1, 4) and perpendicular to PQ is :

(a) $x+y-2z=5$ (b) $x+y-2z=-5$ (c) $x+y+2z=5$ (d) $x+y+2z=-5$

28. Three natural numbers are taken at random from the set $A=\{x:1\le x\le 100, x\in N\}$. The probability that the A.M. of the number taken is 25 is :

(a) $\frac{^{77}C_2}{^{100}C_3}$ (b) $\frac{^{25}C_2}{^{100}C_3}$ (c) $\frac{^{74}C_2}{^{100}C_3}$ (d) $\frac{^{75}C_2}{^{100}C_3}$

29. The mean of five observations is 4 and their variance is 5.2. If three of these observation are 1,2 and 6 then the other two are :

(a) 2 and 9 (b) 3 and 8 (c) 4 and 7 (d) 5 and 6

30. The statement $p\to(q\to p)$ is equivalent to:

(a) $p\to(p\to q)$ (b) $p\to(p\vee q)$ (c) $p\to(p\wedge q)$ (d) $p\to(p\leftrightarrow q)$

ANSWERS

1. (b)	**2.** (a)	**3.** (d)	**4.** (c)	**5.** (c)
6. (b)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (b)
11. (c)	**12.** (c)	**13.** (a)	**14.** (a)	**15.** (b)
16. (b)	**17.** (b)	**18.** (c)	**19.** (b)	**20.** (b)
21. (a)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (b)
26. (a)	**27.** (b)	**28.** (c)	**29.** (c)	**30.** (b)

PRACTICE TEST PAPER - 2

FOR JEE (MAIN)

INSTRUCTIONS :

(i) The question paper consists of '30' objective type questions.

(ii) Each question has four choices (a), (b), (c) and (d) out of which **Only One** is correct.

(iii) You will be awarded **4 marks** for each question, if you have darkened only the bubble corresponding to the correct answer and zero mark if no bubble are darkened. In all other cases, **minus one (–1) mark** will be awarded.

(iv) There is only one correct response for each question. Filling up more than one response in each question will be treated as wrong response and marks for wrong response will be deducted accordingly as per instruction (iii) above.

1. If $f(x) = \max\left(1-|x|, \min(x^2,1)\right)$ then number of point(s) where $f(x)$ is non-differentiable is/are :

(a) 3 (b) 4 (c) 5 (d) 6

2. $\lim_{x\to\infty} \frac{1}{x}\int_0^x \left(\sqrt{t^2+5t}-t\right)dt$ is equal to :

(a) 0 (b) 1 (c) $\frac{5}{2}$ (d) 5

3. If the length of the latus rectum of ellipse is $\frac{5}{2}$ and eccentricity is $\frac{1}{2}$, then the equation of the ellipse in standard form is :

(a) $\frac{x^2}{9}+\frac{y^2}{16}=1$ (b) $\frac{9x^2}{25}+\frac{12y^2}{25}=1$

(c) $\frac{9x^2}{25}+\frac{y^2}{25}=1$ (d) $\frac{x^2}{25}+\frac{12y^2}{25}=1$

4. The mean deviation of the numbers 3, 4, 5, 6, 7 is :

(a) 0 (b) 1.2 (c) 5 (d) 2.5

5. Let $f(x)$ be an odd function defined on R such that $f(1)=2, f(3)=5$ and $f(-5)=$ ...

The value of $\dfrac{f(f(f(-3)))+f(f(0))}{3f(1)-2f(3)-f(5)}$ is :

(a) $\frac{-2}{5}$ (b) $\frac{-2}{3}$ (c) $\frac{2}{5}$ (d) $\frac{2}{3}$

6. If x_1, x_2, x_3, x_4 are positive roots of the equation $x^4-8x^3+ax^2-bx+16=$ ...

then $\tan^{-1}(x_1)+\tan^{-1}(x_2)+\tan^{-1}(x_3)+\tan^{-1}(x_4)$ is equal to :

(a) $\pi+4\tan^{-1}(2)$ (b) $4\tan^{-1}2$

(c) $\pi-4\tan^{-1}2$ (d) $2\tan^{-1}4$

7. If $f(x)$ is a quadratic expression such that $f(-2)=f(2)=0$ and $f(1)=6$, then

$\lim\limits_{x\to0}\dfrac{\sqrt{f(x)}-2\sqrt{2}}{\ln(\cos x)}$ is equal to :

(a) -4 (b) 4 (c) $\frac{1}{\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

8. If the lines joining the origin and points of intersection of $kx+hy=2hk$ and the circle $(x+h)^2+(y-k)^2=a^2$ are perpendicular then h^2+k^2 is equal to :

(a) -1 (b) a (c) a^2 (d) 1

9. Let $f(x)=\begin{cases}\left(\dfrac{(1+2x)^{\frac{1}{x}}}{e^2}\right)^{\frac{1}{x}}, & x\neq0\\ k, & x=0\end{cases}$. If $f(x)$ is continuous at $x=0$, then the value of k is:

(a) 1 (b) e (c) e^2 (d) e^{-2}

10. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$ then angle between $\vec{a}$ and $\vec{b}$ is :

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{4}$

11. Let $f:(0,\infty)\to\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ be a function defined as $f(x)=\tan^{-1}(\ln x)$ and g be the inverse function of f, then $g'(0)$ is equal to :

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) 1 (d) 0

12. A bag contains 4 red and 3 black balls. A second bag contains 2 red and 4 black balls. One bag is selected at random. From the selected bag, one ball is drawn. The probability that the ball drawn to be red, is :

(a) $\frac{13}{42}$ (b) $\frac{14}{42}$ (c) $\frac{19}{42}$ (d) $\frac{23}{42}$

13. If $f(x) = x^3 + 3x + 4$ and g is the inverse function of f then the value of $\frac{d}{dx}\left(\frac{g(x)}{g(g(x))}\right)$ at $x = 4$ equals :

(a) $-\frac{1}{6}$ (b) 6 (c) $-\frac{1}{3}$ (d) 3

14. Number of integral values of 'a' for which the equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ has a solution, is :

(a) 1 (b) 2 (c) 3 (d) 4

15. $\dfrac{\int_0^\infty x^9 e^{-x^2} dx}{\int_0^\infty x^7 e^{-x^2} dx}$ is equal to :

(a) 2 (b) 4 (c) 6 (d) 8

16. The values of parameter such that the line $(\log_2(1 + 5a - a^2))x - 5y - (a^2 - 5) = 0$ is a normal to the curve $xy = 1$, may lie in the interval :

(a) $(-\infty, 0)$ (b) $(5, \infty)$ (c) $(0, 5)$ (d) $(-\infty, 0) \cup (5, \infty)$

17. If r_1, r_2, r_3 are the exradii of a triangle ABC then $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$ is equal to :

(Note: Symbols have their usual meaning in a triangle ABC)

(a) 0 (b) $2S$ (c) S (d) $3r$

18. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x then the largest value which $f(2)$ can attain is :

(a) 7 (b) -7 (c) 13 (d) 8

19. If two tangents are drawn from a point on the circle $x^2 + y^2 + 6x + 6y - 14 = 0$ to the circle $x^2 + y^2 + 6x + 6y + 2 = 0$, then the angle between the tangents, is :

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

20. A pole subtends an angle of 30° at the foot of a tower of height H and at a distance d from the tower, the angle of depression of the foot of the pole from the top of the tower is 60°. The height of the pole is :

(a) $\frac{H}{3}$ (b) $\frac{H}{d}$ (c) $\frac{H}{3d}$ (d) $\frac{3H}{d}$

21. Let $f(x)$ be a polynomial function of degree 4 on R such that $\lim_{x\to 1}\frac{f(x)}{(x-1)^2}=1$. If $f'(0)=-6$ and $f'(2)=6$ then the area bounded by $f(x)$ and co-ordinate axes is :

(a) $\frac{7}{15}$ (b) $\frac{8}{15}$ (c) $\frac{11}{15}$ (d) $\frac{13}{15}$

22. The minimum area of circle which touches the parabolas $y=\frac{1}{2}(x^2+5)$ and $x=\frac{1}{2}(y^2+5)$ is :

(a) π (b) 2π (c) $2\sqrt{2}\pi$ (d) $\sqrt{2}\pi$

23. If a twice differentiable function satisfying a relation $f(x^2y)=x^2f(y)+yf(x^2)$, $\forall x, y>0$ and $f'(1)=1$, then the value of $f''\left(\frac{1}{7}\right)$ is :

(a) 7 (b) $\frac{7}{2}$ (c) $\frac{7}{3}$ (d) $\frac{7}{4}$

24. Let $a_1, a_2, a_3, \ldots\ldots\ldots, a_{10}$ are in G.P. with $a_{51}=25$ and $\sum_{i=1}^{101} a_i=125$, then the value of $\sum_{i=1}^{101}\left(\frac{1}{a_i}\right)$ equals :

(a) 5 (b) $\frac{1}{5}$ (c) $\frac{1}{25}$ (d) $\frac{1}{125}$

25. $\int x^x\left(\frac{1}{x}+ln^2 x+ln x\right)dx$ is equal to :

(a) $x^x\left(ln^2 x-\frac{1}{x}\right)+C$ (b) $x^x(ln x-x)+C$

(c) $\frac{x^x ln^2 x}{2}+C$ (d) $x^x ln x+C$

26. $\sim(p\vee q)\vee(\sim p\wedge q)$ is logically equivalent to :

(a) $\sim p$ (b) p (c) q (d) $\sim q$

27. Let the area enclosed by the curve $y = 1 - x^2$ and the line $y = a$, where $0 \le a \le 1$, be represented by $A(a)$. If $\frac{A(0)}{A\left(\frac{1}{2}\right)} = k$ then :

(a) $1 < k < \frac{3}{2}$ (b) $\frac{3}{2} < k < 2$ (c) $2 < k < \frac{5}{2}$ (d) $\frac{5}{2} < k < 3$

28. The solution of the differential equation $x^2 dy + y(x + y)dx = 0$ is :

(a) $x^2 y = k(2x + y)$ (b) $xy^2 = k(2x + y)$

(c) $x^2 y = k(x + 2y)$ (d) $xy^2 = k(2x + y)$

29. The length of the perpendicular drawn from (1,2,3) to the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is :.

(a) 4 (b) 5 (c) 6 (d) 7

30. If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$ and $x^2 - y^2 = c^2$ cut at right angles, then :

(a) $a^2 + b^2 = 2c^2$ (b) $b^2 - a^2 = 2c^2$ (c) $a^2 - b^2 = 2c^2$ (d) $a^2 b^2 = 2c^2$

ANSWERS

1. (c)	2. (c)	3. (b)	4. (b)	5. (c)
6. (b)	7. (c)	8. (c)	9. (d)	10. (d)
11. (c)	12. (c)	13. (c)	14. (c)	15. (b)
16. (c)	17. (a)	18. (a)	19. (d)	20. (a)
21. (b)	22. (b)	23. (a)	24. (b)	25. (d)
26. (a)	27. (d)	28. (a)	29. (d)	30. (c)

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PRACTICE TEST PAPER - 3

FOR JEE (MAIN)

INSTRUCTIONS :

(i) The question paper consists of '30' objective type questions.

(ii) Each question has four choices (a), (b), (c) and (d) out of which **Only One** is correct.

(iii) You will be awarded **4 marks** for each question, if you have darkened only the bubble corresponding to the correct answer and zero mark if no bubble are darkened. In all other cases, **minus one (–1) mark** will be awarded.

(iv) There is only one correct response for each question. Filling up more than one response in each question will be treated as wrong response and marks for wrong response will be deducted accordingly as per instruction (iii) above.

1. Let u, v and w be three complex numbers such that $|v|=|w|=1$; $|u|<1$ and $w=\dfrac{v(z-u)}{(\bar{z}u-1)}$. Where z be a variable complex number. Locus of $P(z)$ will be :

(a) straight line (b) circle (c) ellipse (d) hyperbola

2. If the equation $x^4-(2x^2+1)a+a^2=0$ has four distinct real roots then the least integral value of a is :

(a) 0 (b) 1 (c) 2 (d) 3

3. In $\Delta ABC, BC=a, AC=b$ and $AB=c$, circumradius of ΔABC is 1. Maximum value of $(ab+bc+ca)$ is :

(a) 3 (b) 5 (3) 7 (d) 9

4. $\displaystyle\int_0^1 \frac{\tan^{-1}x}{\cot^{-1}(1-x+x^2)}\,dx$ equals :

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

5. Let $a_1, a_2, \dots\dots\dots, a_n$ be n positive numbers in G.P. If A is the AM, G is the GM and H is the HM of $a_1, a_2, \dots\dots\dots, a_n$. Then $A+G, 2G$ and $H+G$ are in :

(a) AP (b) GP (c) HP (d) none of these

6. Let A and B be two square matrices of order 3 such that $A^2B=0$. If $BB^T=I$ (I is a unit matrix of order 3) then which of the following may be matrix A ?

(a) $\begin{bmatrix} 2 & 1 & 5 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

7. If the lines $y=x$ and $y=0$ be the tangents of a parabola with focus $(3, 5)$ then length of latus rectum of the parabola is :

(a) $\frac{12}{\sqrt{17}}$ (b) $\frac{20}{\sqrt{17}}$ (c) $\frac{6}{\sqrt{17}}$ (d) $\frac{18}{\sqrt{17}}$

8. Let $a_1, a_2, \dots\dots\dots a_n$ be $n (n \geq 10)$ natural numbers in AP such that volume of parallelopiped formed by vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = a_2\hat{i} + a_3\hat{j} + a_1\hat{k}$ and $\vec{c} = a_3\hat{i} + a_1\hat{j} + a_2\hat{k}$ is 54 then a_6 equals :

(a) 2 (b) 5 (c) 7 (d) 10

9. Let $y = f(x)$ be a function which is discontinuous for exactly 3 values of x but defined $\forall x \in R$. Let $y = g(x)$ is another differentiable function such that $y = f(x) \cdot g(x)$ is continuous $\forall x \in R$. Then minimum number of distinct real roots of the equation $g(x) \cdot g'(x) = 0$ is :

(a) 2 (b) 3 (c) 5 (d) more than 5

10. Let $y = f(x)$ be a function defined as $f(x) = (x-1)(|x^2 - 3x + 2| + |x^2 - 4x + 3|)$ then number of points where $y = f(x)$ is not differentiable is/are :

(a) 0 (b) 1 (c) 2 (d) 3

11. Let $y = f(x)$, be a discontinuous function for exactly 3 distinct positive values of x. Let $g(x) = x^3 + ax^2 + bx + c$. If the function $h(x) = g(x) \cdot f(x)$ is a continuous function then number of points where $y = |g(|x|)|$ is not differentiable is/are :

(a) 3 (b) 4 (c) 5 (d) more than 5

12. $\lim_{x \to 0} \frac{1 - \cos \pi(1 + \cos 2x)}{x^4}$ equals :

(a) $2\pi^2$ (b) $4\pi^2$ (c) $8\pi^2$ (d) 8π

13. Let $f: R \to R$ be a function defined as $f(x) = x^3 - 2x^2 + ax + 5 + 3\sin x + 4\cos x$. If $f(x)$ is a bijective function then the least integral value of a, is :

(a) 6 (b) 7 (c) 5 (d) 8

14. Let $f: A \to B$ and $g: C \to D$ be two bijective functions and $h(x) = f(g(x))$ is defined then $y = h(x)$ must be :

(a) bijective

(b) one-one but we can not say about onto

(c) onto but we can not say about one-one

(d) we can not say anything

15. Let $f(x) = \sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$ be a function monotonic increasing in (a, b) then maximum value of $|a - b|$ is :

(a) 2π (b) π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

16. Let $\vec{a}$ and $\vec{b}$ be two unit vectors then the maximum value of $\dfrac{|\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2}{|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2}$ is equal to :

(a) 1 (b) 2 (c) 4 (d) 6

17. From (0, 0) a tangent is drawn to the curve $y = e^x$ then the area bounded by the tangent, $y = e^x$ and y-axis is :

(a) $2e$ (b) e (c) $\frac{e}{2}$ (d) $\frac{e}{2} - 1$

18. Let $y = f(x)$ be a function defined as $f: R \to R$ such that slope of tangent at $(t, f(t))$ is given by $\dfrac{1}{t + f(t)}$ $\forall t \in R, f(0) = -1$ then $y = f(x)$ must be :

(a) one-one and onto (b) one-one but not onto

(c) onto but not one-one (d) neither one-one nor onto

19. Let $L_1: 2x + 3y = 5$ and L_2 be equal sides of a triangle. If $L: x + y = 2$ be the perpendicular bisector of third side then the equation of L_2 is :

(a) $3x + y = 4$ (b) $x + 5y = 6$ (c) $x = y$ (d) $3x + 2y = 5$

20. If the circles $x^2 + y^2 - 2x - 2y - 50 = 0$ and $x^2 + y^2 - 4x + 2ky + 52 = 0$ cut each other orthogonally then the value of k is :

(a) 1 (b) 2 (c) -1 (4) none of these

21. If $\int x^{23}(1-x)^{31}(3-7x)dx = \dfrac{x^p(1-x)^q}{r} + C$ where $p, q, r \in N$ and C is integration constant then H.C.F. of (p, q, r) is :

(a) 8 (b) 4 (c) 2 (d) 1

22. Let A and B be two independent events such that $P(A)+P(B)=\frac{1}{2}$, $P\left(\overline{A}/\overline{B}\right)=\frac{2}{3}$ then $P\left(\overline{A}\cap B\right)$ equals :

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{9}$ (d) $\frac{1}{18}$

23. Let A and B be two sets such that $n(A)=3$ and $n(B)=4$. Let C be a set such that $C=A\times B$. From set C, the number of ways to select two subsets P and Q with replacement such that $n(P\cap Q)=0$, is :

(a) 2^7 (b) 2^{12} (c) 3^7 (d) 3^{12}

24. Let $y=f(x)$, $y=g(x)$ and $y=h(x)$ be three differentiable function such that $f(x)=x^3+x$, $g(f(x))=x$ and $h(g(g(x)))=x$ then $h'(1)$ is equal to :

(a) 48 (b) 52 (c) 69 (d) 84

25. Number of solutions of the $\sin^8\frac{\pi}{2}x-\cos^2\frac{\pi}{2}x=x^2-4x+5$ in $[-2\pi,2\pi]$, is :

(a) 0 (b) 1 (c) 2 (d) more than 2

26. $\dfrac{(4\cos^2 9^\circ-3)(4\cos^2 27^\circ-3)}{\tan 9^\circ}$ is equal to :

(a) 4 (b) 3 (c) 2 (d) 1

27. In the expansion of $(x^2+1)(x^3+1)(x+1)^5$, the coefficient of x^3 is :

(a) 29 (b) 35 (c) 16 (d) 9

28. In the lines $\frac{x-2}{3}=\frac{y-4}{1}=\frac{z-1}{k}$ and $\frac{x-4}{1}=\frac{y-2}{3}=\frac{z-1}{4}$ be intersecting lines then the length of perpendicular drawn from their point of intersection to the plane $2x-2y+z+4=0$, is :

(a) 3 (b) 1 (c) 5 (d) 9

29. If statement $(p\to q)\to(q\to r)$ is false then truth values of p,q,r respectively, can be :

(a) FTF (b) TTT (c) FFF (d) FTT

30. Number of ways to distribute 15 identical balls among four children such that no child will get more than 5 balls, is :

(a) 455 (b) 56 (c) 816 (d) none of these

ANSWERS

1. (b)	2. (c)	3. (d)	4. (b)	5. (c)
6. (c)	7. (b)	8. (d)	9. (c)	10. (c)
11. (d)	12. (a)	13. (b)	14. (b)	15. (c)
16. (a)	17. (d)	18. (a)	19. (d)	20. (d)
21. (a)	22. (c)	23. (d)	24. (b)	25. (a)
26. (d)	27. (c)	28. (a)	29. (a)	30. (b)

PRACTICE TEST PAPER - 4

FOR JEE (ADVANCED)

INSTRUCTIONS :

(i) The question paper consists of '20' questions.

(ii) **Marking Scheme :**

PART-A

(a) Q. 1 to Q. 8 are **Single Correct** type questions. Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct. For each question you will be awarded **3 marks.** In all other cases, **minus one (–1) mark** will be awarded.

(b) Q. 9 to Q. 14 are **Paragraph** type questions. Based upon the paragraph 3 single correct type questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct. For each question, you will be awarded **4 marks.** In all other cases, **minus one (–1) mark** will be awarded.

(c) Q. 15 to Q. 18 are **Multiple Correct** type questions. Each question has four choices (a), (b), (c) and (d) out of which **ONE OR MORE** may be correct. There is a partial marking, **1 mark** for each correct option, provided **NO** incorrect option is darkened and **4 marks** for all correct. For each question you will be awarded 4 **marks.** In all other cases, **minus one (–1) mark** will be awarded.

PART-B

(d). Q. 1 to Q. 2 are **Matrix** type questions. Each question has **four** statements (a, b, c, d) given in Column-1 and **five** statments (p, q, r, s, t) given in Column-II. Any given statement in Column-I can have correct matching with one or more statement(s) given in Column-II. For each question, you will be awarded **2 marks** for each row, thus each question carries a maximum of **8 marks**. No negative marks will be awarded in this section.

PART-A

[SINGLE CORRECT CHOICE TYPE]

Q. 1 to Q. 8 has four choices (a), (b), (c), (d) out of which ONLY ONE is correct.

1. If equation of a plane parallel to y-axis and containing the points (1, 0, 1) and (3, 2, –1) is $ax + by + cz = 2$, then the value of $|a + b + c|$ is :

(a) 1 (b) 2 (c) 3 (d) 4

2. If $f'(\theta) = \sin^2\theta|\cos\theta| \forall \theta \in R$ and $f\left(\frac{2\pi}{3}\right) = 1 - \frac{\sqrt{3}}{8}$ then the value of $\lim_{\theta \to \frac{3\pi^-}{2}} f(\theta)$ is :

(a) $\frac{-1}{3}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $\frac{-2}{3}$

3. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all roots x_1, x_2, x_3 and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. The smallest value of the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$, is :

(a) 16 (b) 8 (c) 12 (d) 24

4. A circle is inscribed into a rhombus $ABCD$ with an angle of 60°. The distance from the centre of the circle to the nearest vertex is equal to 1. If P be any point of the circle, then $|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$ is equal to :

(a) 11 (b) 9 (c) 7 (d) none of these

5. Suppose $g(x)$ is continuous on [–1, 1] with $g(-1) = -1$ and $g(1) = 1$. Which one of the following must be true ?

(a) There is value of c in $(-1, 1)$ where $g(c)$ equals –1 or 1

(b) There is a unique value of c in $(-1, 1)$ where $g(c) = \frac{1}{2}$

(c) There is a value of c in $(-1, 1)$ where $g(c)$ equals the area of a circle with radius $\frac{1}{2}$

(d) None of the above

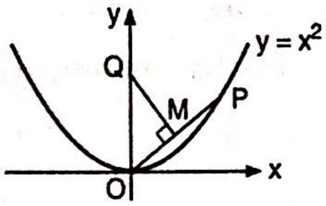
6. Let a line with the inclination angle of 60° be drawn through the focus F of the parabola $y^2 = 8(x + 2)$. If the two intersection points of the line and the parabola are A and B, and the perpendicular bisector of the chord AB intersects the x-axis at the point P, then the length of the segment PF is :

(a) $\frac{16}{3}$ (b) $\frac{8}{3}$ (c) $\frac{16\sqrt{3}}{3}$ (d) $8\sqrt{3}$

7. The value of the definite integral $\int_{\pi/8}^{3\pi/8} \frac{11+4\cos 2x+\cos 4x}{1-\cos 4x}\,dx$ equals :

(a) $-6-\frac{\pi}{4}$ (b) $6\sqrt{2}-\frac{\pi}{4}$ (c) $12-\frac{\pi}{2}$ (d) $6-\frac{\pi}{4}$

8. The figure shows a point P on the parabola $y=x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As the point P approaches the origin along the parabola, the limiting ordinate of Q is :



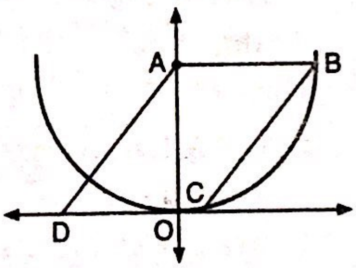
(a) 1/4 (b) 1/2
(c) 2/3 (d) 1

[PARAGRAPH TYPE]

Q. 9 to Q. 14 has four choices (a), (b), (c), (d) out of which ONLY ONE is correct.

Paragraph for Question no. 9 to 11

Let O be the origin and A be the focal point of the parabola $P_1: y=kx^2\,(k>0)$. Let $ABCD$ be a rhombus with CD lying on the x-axis and B lying on the parabola P_1. Also area of $ABCO$ is $4-\sqrt{3}$ and $P_2: y^2=4x$.



9. The value of k is equal to :

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{4\sqrt{2}}$ (c) $2\sqrt{2}$ (d) $4\sqrt{2}$

10. Area bounded by the curves P_1 and P_2 is :

(a) $\frac{4\sqrt{2}}{3}$ (b) $\frac{8\sqrt{2}}{3}$ (c) $\frac{16\sqrt{2}}{3}$ (d) $\frac{32\sqrt{2}}{3}$

11. If the line $y=\frac{1}{\sqrt{2}}x+c$ is normal to parabola P_1 then the value of c is equal to :

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $3\sqrt{2}$ (d) $4\sqrt{2}$

Paragraph for Question no. 12 to 14

Let f and g are two linear functions such that $f(x-1)=2x-3+g(1)-f(1)$ and $g(x-1)=4x+5-g(1)-f(1)$ for all real number x.

12. The value of $\lim_{x\to 0}\left(\frac{g(x)}{f(x)}\right)^{\text{cosec}\,x}$ is equal to :

(a) e (b) e^2 (c) e^3 (d) e^4

13. If $\int \frac{f(x)+2}{\left(\frac{f(x)-1}{2}\right)\left(\frac{f(x)+1}{2}\right)\left(\frac{g(x)+7}{4}\right)\left(\frac{g(x)+11}{4}\right)+1}dx = C - \frac{1}{h(x)}$ where $h(x)$ is of the form of $ax^2 + bx + c$, then $(a+b+c)$ equals :

(a) 4 (b) 5 (c) 6 (d) none of these

14. Area enclosed by the curves $y = f(x)$, $y = g(x)$ and $y = 0$, is :

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) 1

[MULTIPLE CORRECT CHOICE TYPE]

Q. 15 to Q. 18 has four choices (a), (b), (c), (d) out of which **ONE OR MORE** may be correct.

15. Let $z \in C$ and z_1, z_2, z_3 be the vertices of an equilateral triangle. Then :

(a) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

(b) $\frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3} = 0$ where $z = \frac{z_1 + z_2 + z_3}{3}$

(c) $(z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$ where $\omega = e^{\frac{i2\pi}{3}}$ and $i = \sqrt{-1}$

(d) $\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix} = 0$

16. Assume that $M = \{(x, y) \mid x^2 + 2y^2 = 3\}$ and $N = \{(x, y) \mid y = mx + b\}$. If $M \cap N \neq \phi$ for all $m \in R$, then b can be :

(a) -2 (b) -1 (c) $\frac{-\sqrt{6}}{2}$ (d) $\frac{\sqrt{6}}{2}$

17. For the 3 events A, B and C, P (atleast one occurring) $= \frac{3}{4}$, P (atleast two occurring) $= \frac{1}{2}$ and P (exactly two occurring) $= \frac{2}{5}$. Which of the following relations is / are **CORRECT** ?

(a) $P(ABC) = \frac{1}{10}$ (b) $P(AB) + P(BC) + P(CA) = \frac{7}{5}$

(c) $P(A) + P(B) + P(C) = \frac{27}{20}$ (d) $P(A\overline{B}\overline{C}) + P(\overline{A}B\overline{C}) + P(\overline{A}\overline{B}C) = \frac{1}{4}$

18. Let $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$, then the correct statement is :

(a) $A^2-4A-5I_3=0$

(b) $A^{-1}=\frac{1}{5}(A-4I_3)$

(c) A^3 is not invertible

(d) A^2 is invertible

PART-B

[MATRIX TYPE]

Q. 1 & Q. 2 has four statements (a, b, c, d) given in **Column-I** and five statements (p, q, r, s, t) given in **Column-II**. Any given statement in **Column-I** can have correct matching with one or more statement(s) given in **Column-II**.

1.

	Column-I	Column-II	
(a)	Let S_1 and S_2 are two points on AB of a ΔABC with vertices $A(-2,3)$, $B(4,-6)$ and $C(1,1)$. CS_1 and CS_2 divide the triangle into three parts of equal area. If the equation of the lines through the origin drawn parallel to CS_1 and CS_2 is $y^2+\lambda xy-\mu x^2=0$ then the value of $(\lambda+\mu)$, is	(p)	2
(b)	If the area enclosed by the curves $y=\|x\|$ and $x^2+y-2=0$ is $\frac{p}{q}$, where p and q are coprime then the value of $(p-q)$, is	(q)	4
(c)	The curve that satisfies the differential equation $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$ is a hyperbola passing through (2, 1) with eccentricity e. The value of e^4 is equal to	(r)	6
(d)	Let $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{ln(n^2+r^2)-2n\,ln\,n}{n}=ln\,2+\frac{\pi}{2}-2$	(s)	7
	If $\lim_{n\to\infty}\frac{1}{n^{2m}}[(n^2+1^2)^m(n^2+2^2)^m.......(n^2+n^2)^m]^{\frac{1}{n}}=\left(\frac{a\sqrt{e^{\pi}}}{e^2}\right)^m$, then the value of a, is	(t)	8

2.

	Column-I	Column-II	
(a)	If the distance of the plane passing through the point $P(1,1,1)$ and perpendicular to the line $\frac{x-1}{3}=\frac{y-1}{0}=\frac{z-1}{4}$ from the origin is $\frac{p}{q}$ (where p and q are coprime), then the value of $(p-q)$ is	(p)	1
(b)	The interval [0,4] is divided into n equal sub-intervals by the points $x_0, x_1, x_2 ..., x_{n-1}, x_n$ where $0 = x_0 < x_1 < x_2 < x_3 .... < x_n = 4$. If $\delta x = x_i - x_{i-1}$ for $i = 1,2,3,....n$ then the value of $\lim_{\delta x \to 0} \sum_{i=1}^{n} x_i \delta x$ is	(q)	2
(c)	Let $f: R \to R$ be a function satisfying $xf(x) + (1-x)f(-x) = x^2 + x + 1$ for any real number x. The greatest real number M for which $f(x) \geq M$ for all real numbers x, is equal to $\frac{p}{q}$, where p and q are coprime. The value of $(q-p)$, is	(r)	3
(d)	The area of the trapezium $ABCD$ with $AB \| CD, AD \perp AB$ and $AB = 3CD$ is equal to 4. A circle inside the trapezium is tangent to all of its sides. If the radius of the circle is r then the value of $4r^2$, is	(s)	6
		(t)	8

ANSWERS

PART-A

Single Correct Choice Type

1. (b) **2.** (b) **3.** (a) **4.** (a) **5.** (c) **6.** (a) **7.** (d) **8.** (b)

Paragraph Type

9. (b) **10.** (c) **11.** (d) **12.** (b) **13.** (b) **14.** (c)

Multiple Correct Choice Type

15. (a, b, c, d) **16.** (b, c, d) **17.** (a, c, d) **18.** (a, b, d)

PART-B

Matrix Type

1. (a) (s); (b) (q); (c) (q); (d) (p)
2. (a) (q); (b) (t); (c) (p); (d) (r)

❏❏❏

PRACTICE TEST PAPER - 5

FOR JEE (ADVANCED)

INSTRUCTIONS :

(i) The question paper consists of '**19**' questions.

(ii) **Marking Scheme :**

PART-A

(a) Q. 1 to Q. 4 are **Single Correct** type questions. Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct. For each question you will be awarded **3 marks.** In all other cases, **minus one (–1) mark** will be awarded.

(b) Q. 5 to Q. 9 are **Multiple Correct** type questions. Each question has four choices (a), (b), (c) and (d) out of which **ONE OR MORE** may be correct. For each question you will be awarded **4 marks.** In all other cases, **minus one (–1) mark** will be awarded.

PART-B

(c). Q. 1 to Q. 2 are **Matrix** type questions. Each question has **four** statements (a, b, c, d) given in Column-1 and **five** statments (p, q, r, s, t) given in Column-II. Any given statement in Column-I can have correct matching with one or more statement(s) given in Column-II. For each question, you will be awarded **2 marks** for each row, thus each question carries a maximum of **8 marks**. There is no negative marks awarded for incorrect answer(s) in this section.

PART-C

(d) Q. 1 to Q. 8 are **Integer** type questions. The answer to each question is a **single digit integer** ranging from 0 to 9. For each question, you will be awarded **4 marks**. In case of Integer questions, round off answer to nearest integer. No negative marks will be awarded in this section.

PART-A

[SINGLE CORRECT CHOICE TYPE]

Q. 1 to Q. 4 has four choices (a), (b), (c), (d) out of which **ONLY ONE** is correct.

1. If $\int_0^{4\pi} \ln|13\sin x + 3\sqrt{3}\cos x|\,dx = k\pi\ln 7$, then the value of k is :

(a) 2 (b) 4 (c) 8 (d) 16

2. If the variable line $y = kx + 2h$ is a tangent to an ellipse $2x^2 + 3y^2 = 6$ then locus of $P(h, k)$ is a conic C whose eccentricity equals :

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{7}}{3}$ (c) $\sqrt{\frac{7}{3}}$ (d) $\sqrt{2}$

3. If complex number z_1 satisfies $\arg z = \frac{\pi}{4}$ and $|z - 2 - i| = 1$ and z_2 is the complex number which lies on the curve $|z - 25i| = 15$ & having least positive argument then minimum value of $|z_1 - z_2|$ will be :

(a) $\sqrt{346}$ (b) $3\sqrt{74}$ (c) $2\sqrt{74}$ (d) $\sqrt{276}$

4. If p, q, r are prime numbers and α, β, γ are positive integers such that L.C.M. of α, β, γ is p^3q^2r and greatest common divisor of α, β, γ is pqr, then the number of possible triplets (α, β, γ) will be :

(a) 6 (b) 12 (c) 72 (d) 96

[MULTIPLE CORRECT CHOICE TYPE]

Q. 5 to Q. 9 has four choices (a), (b), (c), (d) out of which **ONE OR MORE** may be correct.

5. Two soldiers X and Y shoot each other. If one or both are hit then shooting is over. If both shot miss then they repeat the process. Suppose the results of shots are independent and that each shot of X will hit Y with probability $\frac{2}{5}$ and each shot of Y will hit X with probability $\frac{1}{5}$.

(a) The probability that X is not hit is $\frac{8}{13}$

(b) The probability that X is not hit is $\frac{8}{25}$

(c) The probability that both are hit is $\frac{2}{25}$

(d) The probability that both are hit is $\frac{2}{13}$

6. If $P(x) = \text{mid.}\,(g(x), h(x), f(x))$ means the function will be second in order when values of the three functions are arranged at a particular x.

If $P(x) = \text{mid.}\left(x-1, (x-3)^2, 3-\frac{(x-3)^2}{2}\right)$, $x \in [1,3]$ then for given interval :

(a) Number of points of discontinuity of $P(x)$ will be 0

(b) Number of points of discontinuity of $P(x)$ will be 1

(c) Number of points of non-derivability of $P(x)$ will be 1

(d) Number of points of non-derivability of $P(x)$ will be 2

7. Let $P(x)$ be a polynomial of least degree with leading coefficient 1 and rational coefficients, such that $P\left(\sqrt{5+\sqrt{5+\sqrt{5+\ldots\ldots\ldots\infty \text{ terms}}}}\right) = 0$ then :

(a) the value of $100\sum_{r=2}^{100}\frac{1}{P(r)+5}$ is equal to 99

(b) $\lim_{n\to\infty}\sum_{r=3}^{n}\frac{1}{P(r)+3}$ is equal to 1

(c) $\int_0^1 \frac{dx}{P(x)+x+6}$ is equal to $\frac{\pi}{4}$

(d) $\int_1^2 \frac{dx}{\sqrt{P(x)+x+4}}$ is equal to $ln(2+\sqrt{3})$

8. For $x \in \left(0, \frac{\pi}{4}\right)$

$R_n = \sum_{r=1}^{2n}\sin(\sin^{-1} x^{4r-3})$, $S_n = \sum_{r=1}^{2n}\cos(\cos^{-1} x^{4r-2})$,

$T_n = \sum_{r=1}^{2n}\tan(\tan^{-1} x^{4r-1})$, $U_n = \sum_{r=1}^{2n}\cot(\cot^{-1} x^{4r})$

where $n \in N$ and $n \ge 4$

(a) $R_n < S_n < T_n < U_n$

(b) $R_n > S_n > T_n > U_n$

(c) $\lim_{n\to\infty}(R_n + S_n + T_n + U_n)$ is equal to $\frac{x}{1-x}$

(d) The value of x for which $R_n + S_n = T_n + U_n$ is $2\sin\frac{\pi}{10}$

9. Let P be the plane consisting of all points that are equidistant from $A(-4,2,1)$ and $B(2,-4,3)$ and Q denotes the plane $x - y + Cz = 1$, where $C \in R$.

(a) If $C = \frac{1}{3}$ then plane P will be parallel to Q

(b) If $C = -1$ then plane P will be perpendicular to Q

(c) Volume of tetrahedron formed by plane P with coordinate planes will be $\frac{2}{81}$.

(d) If line $L: \frac{x-1}{1} = \frac{y+2}{3} = \frac{z-7}{-1}$ intersect plane P at $R(\alpha, \beta, \gamma)$, then $\alpha + \beta + \gamma = 12$

PART-B

[MATRIX TYPE]

Q. 1 & Q. 2 has **four** statements (a, b, c, d) given in **Column-I** and **five** statements (p, q, r, s, t) given in **Column-II**. Any given statement in **Column-I** can have correct matching with one or more statements given in **Column-II**.

1.

	Column-I		Column-II
(a)	Total number of values of x, of the form $\frac{1}{n}, n \in N$ in the interval $\left[\frac{1}{15}, \frac{1}{10}\right]$ which satisfies $\{x\} + \{2x\} + \ldots\ldots + \{12x\} = 78x$, (where { } denotes fractional part of x)	(p)	1
(b)	Let $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$ then the value of $\frac{a_9}{a_{11}}$ is equal to	(q)	2
(c)	Let G be centroid of ΔABC whose side length are a, b, c. If P is a point in the plane of ΔABC such that $PA = 1, PB = 2, PC = 1$ & $PG = 1$ then the value of $\frac{a^2 + b^2 + c^2}{9}$ is equal to	(r)	3
(d)	If a derivable function $f: R^+ \to R$ satisfies $f(xy) = f(x) + f(y) \forall\, x, y \in R^+$. If $f(16) = 3$ then $\frac{1}{f(2)} + \frac{1}{f(4)}$ is equal to	(s)	4
		(t)	10

2.

	Column-I		Column-II
(a)	If $A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}, x \neq (2n+1)\frac{\pi}{2}$, then the minimum value of $\lvert A^T A^{-1}\rvert$ is equal to	(p)	0
(b)	If $a \in (a_1, a_2)$ satisfy the condition that the point of local minimum and point of local maximum is less than 4 and greater than –2 respectively for $f(x)=x^3-3ax^2+3(a^2-1)x+1$ then a_2-a_1 is equal to	(q)	1
(c)	If $y=3x-8$ is tangent at (7, 13) on a parabola with focus $(-1,-1)$ and length of latus rectum is l, then $[l]$ is equal to	(r)	2
(d)	A hyperbola has centre at origin and one focus at (6, 8). If its two directrices are $3x+4y+10=0$ and $3x+4y-10=0$ and eccentricity is e, then the value of $\frac{4e^2}{5}$ is equal to	(s)	3
		(t)	4

[Note : $[k]$ denotes greatest integer less than or equal to k.]

PART-C

[INTEGER TYPE]

Q. 1 to Q. 8 are "Integer Type" questions. (The answer to each of the questions are **Single digits.**

1. Let $\int_0^a \frac{\cot^{-1}(e^x)+\cot^{-1}(e^{-x})}{\tan^{-1}a+\tan^{-1}x}\cdot\frac{dx}{x^2+1}=\frac{\pi}{p}\ln q, (a>0)$ where $p, q \in N$, then find the minimum value of $(p+q)$.

2. Let z_1 and z_2 be two complex numbers such that $|z_2|=1$ and $\frac{z_1-1}{z_1+1}=\left(\frac{z_2-1}{z_2+1}\right)^2$, then find the value of $|z_1-1|+|z_1+1|$.

3. Let $f(x)$ be a polynomial satisfies the relation $f(f(f(x)))+(1-p)f(x)=3, \forall x \in R$, where p is any real number. If leading coefficient of $f(x)$ is 2, then find the value of $\left(\frac{d}{dx}(f(f(x)))\text{ at } x=p\right)+\int_{-p}^{p}\left(2f^{-1}(x)+1\right)dx$.

4. A 3×3 determinant has entries either 1 or -1. Let S be the set of all determinants such that the product of elements of any row or column is -1. For example $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$ is an element of S and number of elements in S is m.

Let $P = \begin{bmatrix} 3 & -2 & 3 \\ 2 & -2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$ and trace of the matrix $\text{adj}(\text{adj } P)$ is n then find the value of $\frac{m}{n}$.

5. An ellipse has major & minor axes of length $\sqrt{3}$ and 1 respectively, slides along the coordinate axes and always remains confined in the first quadrant. The locus of centre of ellipse is a circle. Find the number of tangents to the director circle of this circle which are normal to ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

6. Given two intersecting circles C_1 and C_2 with centres at M and N respectively. From one of the point of intersection A, tangents are drawn to two circles meeting the circles at B and C respectively. Let P be any point located such that $PMAN$ is a parallelogram and if $AB + BC + CA = k(PA \sin A + PB \sin B + PC \sin C), k \in R$, then find k. (where angles A, B, C are angles of ΔABC)

7. If a, b, c, p, q and $r \in R - \{0\}$ such that $ap + bq + cr + \sqrt{(a^2 + b^2 + c^2)(p^2 + q^2 + r^2)} = 0$ then find the value of $\frac{aq}{bp} + \frac{br}{cq} + \frac{cp}{ar}$.

8. If a, b, c are sides of triangle ABC and a^3, b^3, c^3 are roots of $x^3 - px^2 + qx - r = 0$ where $p, q, r \in N$ and $\sin^3 A + \sin^3 B + \sin^3 C - 3 \sin A \sin B \sin C = 8\Delta^3$. (where Δ is area of triangle ABC) then find minimum value of $(p + r)$.

ANSWERS

PART-A

Single Correct Choice Type

1. (b) **2.** (c) **3.** (c) **4.** (c)

Multiple Correct Choice Type

5. (a, d) **6.** (a, d) **7.** (a, c, d) **8.** (b, c) **9.** (a, d)

PART-B

Matrix Type

1. (a) (r), (b) (q), (c) (p), (d) (q)

2. (a) (q), (b) (t), (c) (r), (d) (t)

PART-C

Integer Type

1. (4) **2.** (2) **3.** (4) **4.** (8) **5.** (0) **6.** (2) **7.** (3) **8.** (5)

□□□